

THE EXTREMAL PROCESS OF BRANCHING BROWNIAN MOTION.

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ABSTRACT. We prove that the extremal process of branching Brownian motion, in the limit of large times, converges weakly to a cluster point process. The limiting process is a (randomly shifted) Poisson cluster process, where the positions of the clusters is a Poisson process with exponential density. The law of the individual clusters is characterized as branching Brownian motions conditioned to perform "unusually large displacements", and its existence is proved. The proof combines three main ingredients. First, the results of Bramson on the convergence of solutions of the Kolmogorov-Petrovsky-Piscounov equation with general initial conditions to standing waves. Second, the integral representations of such waves as first obtained by Lalley and Sellke in the case of Heaviside initial conditions. Third, a proper identification of the tail of the extremal process with an auxiliary process, which fully captures the large time asymptotics of the extremal process. The analysis through the auxiliary process is a rigorous formulation of the *cavity method* developed in the study of mean field spin glasses.

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1. INTRODUCTION

Branching Brownian Motion (BBM) is a continuous-time Markov branching process which plays an important role in the theory of partial differential equations [5, 6, 26, 29], in the theory of disordered systems [11, 22], and in biology [23]. It is constructed as follows.

Start with a single particle which performs standard Brownian Motion $x(t)$ with $x(0) = 0$, which it continues for an exponential holding time T independent of x , with $\mathbb{P}[T > t] = e^{-t}$. At time T , the particle splits independently of x and T into k offsprings with probability p_k , where $\sum_{k=1}^{\infty} p_k = 1$, $\sum_{k=1}^{\infty} k p_k = 2$, and $K \equiv \sum_k k(k-1)p_k < \infty$. These particles continue along independent Brownian paths starting at $x(T)$, and are subject to the same splitting rule, with the effect that the resulting tree \mathfrak{X} contains, after an elapsed time $t > 0$, $n(t)$ particles located at $x_1(t), \dots, x_{n(t)}(t)$, with $n(t)$ being the random number of particles generated up to that time (it holds that $\mathbb{E}n(t) = e^t$).

The link between BBM and partial differential equations is provided by the following observation due to McKean [29]: if one denotes by

$$u(t, x) \equiv \mathbb{P} \left[\max_{1 \leq k \leq n(t)} x_k(t) \leq x \right] \quad (1.1)$$

the law of the maximal displacement, a renewal argument shows that $u(t, x)$ solves the Kolmogorov-Petrovsky-Piscounov or Fisher [F-KPP] equation,

$$\begin{aligned} u_t &= \frac{1}{2} u_{xx} + \sum_{k=1}^{\infty} p_k u^k - u, \\ u(0, x) &= \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases} \end{aligned} \quad (1.2)$$

The F-KPP equation admits traveling waves: there exists a unique solution satisfying

$$u(t, m(t) + x) \rightarrow \omega(x) \quad \text{uniformly in } x \text{ as } t \rightarrow \infty, \quad (1.3)$$

with the centering term given by

$$m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t, \quad (1.4)$$

and $\omega(x)$ the distribution function which solves the o.d.e.

$$\frac{1}{2} \omega_{xx} + \sqrt{2} \omega_x + \omega^2 - \omega = 0. \quad (1.5)$$

If one excludes the trivial cases, solutions to (1.5) are unique up to translations: this will play a crucial role in our considerations.

Lalley and Sellke [27] provided a characterization of the limiting law of the maximal displacement in terms of a *random shift* of the Gumbel distribution. More precisely, denoting by

$$Z(t) \equiv \sum_{k=1}^{n(t)} \left(\sqrt{2}t - x_k(t) \right) \exp -\sqrt{2} \left(\sqrt{2}t - x_k(t) \right), \quad (1.6)$$

the so-called *derivative martingale*, Lalley and Sellke proved that $Z(t)$ converges almost surely to a strictly positive random variable Z , and established the integral representation

$$\omega(x) = \mathbb{E} \left[\exp \left(-C Z e^{-\sqrt{2}x} \right) \right], \quad (1.7)$$

for some $C > 0$.

It is also known (see e.g. Bramson [12] and Harris [24]) that

$$\lim_{x \rightarrow \infty} \frac{1 - \omega(x)}{x e^{-\sqrt{2}x}} = C. \quad (1.8)$$

(The reason why the two constants C in (1.7) and (1.8) are equal is apparent in [27], and [4]).

Despite the precise information on the maximal displacement of BBM, an understanding of the full statistics of the largest particles is still lacking. The statistics of such particles are fully encoded in the extremal process, the random measure:

$$\mathcal{E}_t \equiv \sum_{k \leq n(t)} \delta_{x_k(t) - m(t)}. \quad (1.9)$$

Few papers have addressed so far the large time limit of the extremal process of branching Brownian motion.

On the physical literature side, we mention the contributions by Brunet and Derrida [18, 19], who reduce the problem of the statistical properties of particles "at the edge" of BBM to that of identifying the finer properties of the delay of travelling waves. (Here and below, "edge" stands for the set of particles which are at distances of order one from the maximum).

On the mathematical side, properties of the large time limit of the extremal process have been established in two papers of ours [3, 4]. In a first paper we obtained a precise description of the *paths* of extremal particles which in turn imply a somewhat surprising restriction of the correlations of particles at the edge of BBM. These results were instrumental in our second paper on the subject where we proved that a certain process obtained by a correlation-dependent thinning of the extremal particles converges to a random shift of a Poisson Point Process (PPP) with exponential density.

It is the purpose of this paper to complete this picture and to provide an explicit construction of the extremal process of branching Brownian motion in the limit of large times. We prove that the limit is a randomly shifted Poisson cluster process. Up to a realization-dependent shift, this point process corresponds to the superposition of independent point processes or *clusters*. The maxima of these point processes, or *cluster-extrema*, form a Poisson point process with exponential density. Relative to their maximum, the laws of the individual clusters are identical. The law of the clusters coincides with that of a branching Brownian motion conditioned to perform "unusually high jumps". The precise statement is given in Section 2.

Understanding the extremal process of BBM is a longstanding problem of fundamental interest. Most results concerning extremal processes of correlated random variables

concern criteria that show that it behaves as if there were no correlations [28]. Bramson's result shows that this cannot be the case for BBM. A class of models where a more complex structure of *Poisson cascades* was shown to emerge are the *generalized random energy models* of Derrida [17, 9]. These models, however, have a rather simple hierarchical structure involving a finite number of levels only which greatly simplifies the analysis, which cannot be carried over to models with infinite levels of branching such as BBM or the *continuous random energy models* studied in [11]. BBM is a case right at the borderline where correlations just start to effect the extremes and the structure of the extremal process. Our results thus allow to peek into the world beyond the simple Poisson structures and hopefully open the gate towards the rigorous understanding of complex extremal structures. Mathematically, BBM offers a spectacular interplay between probability and non-linear pdes, as was noted already by McKean. We will heavily rely on this dual way to attack and understand this problem.

The remainder of this paper is organised as follows. In Section 2 we state our main results, the heuristics behind them, and we indicate the major steps in the proof. In Section 3 we give the details of the proofs.

2. MAIN RESULT

We first recall from [25] the standard infrastructure for the study of point process. Let \mathcal{M} be the space of Radon measure on \mathbb{R} . Elements of \mathcal{M} are in correspondence with the positive linear functionals on $\mathcal{C}_c(\mathbb{R})$, the space of continuous functions on \mathbb{R} with compact support. In particular, any element of \mathcal{M} is locally finite. The space \mathcal{M} is endowed with the vague topology (or weak- \star -topology), that is, $\mu_n \rightarrow \mu$ in \mathcal{M} if and only if for any $\phi \in \mathcal{C}_c(\mathbb{R})$, $\int \phi d\mu_n \rightarrow \int \phi d\mu$. The law of a random element Ξ of \mathcal{M} , or random measure, is determined by the collection of real random variables $\int \phi d\Xi$, $\phi \in \mathcal{C}_c(\mathbb{R})$. A sequence (Ξ_n) of random elements of \mathcal{M} is said to converge to Ξ if and only if for each $\phi \in \mathcal{C}_c(\mathbb{R})$, the random variables $\int \phi d\Xi_n$ converges in the weak sense to $\int \phi d\Xi$. A point process is a random measure that is integer-valued almost surely. It is a standard fact that point processes are closed in the set of random elements of \mathcal{M} .

The limiting point process of BBM is constructed as follows. Let Z be the limiting derivative martingale. Conditionally on Z , we consider the Poisson point process (PPP) of density $CZ\sqrt{2}e^{-\sqrt{2}x}dx$:

$$P_Z \equiv \sum_{i \in \mathbb{N}} \delta_{p_i} \equiv \text{PPP} \left(CZ\sqrt{2}e^{-\sqrt{2}x}dx \right), \quad (2.1)$$

with C as in (1.7). Now let $\{x_k(t)\}_{k \leq n(t)}$ be a BBM of length t . Consider the point process of the gaps $\sum_k \delta_{x_k(t) - \max_j x_j(t)}$ conditioned on the event $\{\max_j x_j(t) - \sqrt{2}t > 0\}$. Remark that, in view of (1.4), the probability that the maximum of BBM shifted by $-\sqrt{2}t$ does not drift to $-\infty$ is vanishing in the large time limit. In this sense, the BBM is conditioned to perform "unusually large displacements". It will follow from the proof that the law of this process exists in the limit. Write $\mathcal{D} = \sum_j \delta_{\Delta_j}$ for a point process with this law and consider independent, identically distributed (iid) copies $(\mathcal{D}^{(i)})_{i \in \mathbb{N}}$. The main result states that the extremal process of BBM as a point process converges as follows:

Theorem 2.1 (Main Theorem). *Let P_Z and $\mathcal{D}^{(i)} = \{\Delta_j^{(i)}\}_{j \in \mathbb{N}}$ be defined as above. Then the family of point processes \mathcal{E}_t , defined in (1.9), converges in distribution to a point process, \mathcal{E} , given by*

$$\mathcal{E} \equiv \lim_{t \rightarrow \infty} \mathcal{E}_t \stackrel{\text{law}}{=} \sum_{i,j} \delta_{p_i + \Delta_j^{(i)}}. \quad (2.2)$$

We remark in passing that a similar structure is expected to emerge in all the models which are conjectured to fall into the universality class of branching Brownian motion, such as the *2-dim Gaussian Free Field* [7, 8, 14], or the *cover time* for the simple random walk on the two dimensional discrete torus [20, 21]. Loosely, the picture which is expected in all such models is that of "fractal-like clusters well separated from each other".

The key ingredient in the proof of Theorem 2.1 is an identification of the extremal process of BBM with an auxiliary process constructed from a Poisson process, with an explicit density of points in the tail. This is a rigorous implementation of the *cavity approach* developed in the study of mean field spin glasses [30] for the case of BBM, and might be of interest to determine extreme value statistics for other processes. We discuss the idea of the proof in the next section.

2.1. The Laplace transform of the extremal process of BBM. Recently, Brunet and Derrida [19] have shown the existence of statistics of the extremal process \mathcal{E}_t in the limit of large times. We prove here that the limit of \mathcal{E}_t exists as a point process using the convergence of the Laplace functionals,

$$\Psi_t(\phi) \equiv \mathbb{E} \left[\exp \left(- \int \phi(y) \mathcal{E}_t(dy) \right) \right], \quad (2.3)$$

for $\phi \in \mathcal{C}_c(\mathbb{R})$ non-negative. This gives the existence result in the proof of the main theorem.

Proposition 2.2 (Existence of the limiting extremal process). *The point process \mathcal{E}_t converges in law to a point process \mathcal{E} .*

It is easy to see that the Laplace functional is a solution of the F-KPP equation following the observation of McKean, see Lemma 3.1 below. However, convergence is more subtle. It will follow from the convergence theorem of Bramson, see Theorem 3.2 below, but only after an appropriate truncation of the functional needed to satisfy the hypotheses of the theorem. The proof recovers a representation of the form (1.7). More importantly, we obtain an expression for the constant C as a function of the initial condition. This observation is inspired by the work of Chauvin and Rouault [15]. It will be at the heart of the representation theorem of the extremal process as a cluster process.

Proposition 2.3. *Let \mathcal{E}_t be the process (1.9). For $\phi \in \mathcal{C}_c(\mathbb{R})$ non-negative,*

$$\lim_{t \rightarrow \infty} \mathbb{E} \left[\exp \left(- \int \phi(y+x) \mathcal{E}_t(dy) \right) \right] = \mathbb{E} \left[\exp \left(-C(\phi) Z e^{-\sqrt{2}x} \right) \right] \quad (2.4)$$

where, for $u(t, y)$ solution of F-KPP with initial condition $u(0, y) = e^{-\phi(y)}$,

$$C(\phi) = \lim_{t \rightarrow \infty} \sqrt{\frac{2}{\pi}} \int_0^\infty \left(1 - u(t, y + \sqrt{2}t)\right) y e^{\sqrt{2}y} dy$$

is a strictly positive constant depending on ϕ only, and Z is the derivative martingale.

A straightforward consequence of Proposition 2.3 is the *invariance under superpositions* of the random measure \mathcal{E} , conjectured by Brunet and Derrida [18, 19].

Corollary 2.4 (Invariance under superposition). *The law of the extremal process of the superposition of n independent BBM started in $x_1, \dots, x_n \in \mathbb{R}$ coincides in the limit of large time with that of a single BBM, up to a random shift.*

As conjectured in [19], the superposability property is satisfied by any cluster point process constructed from a Poisson point process with exponential density to which, at each atom, is attached an iid point process. Theorem 2.5 shows that this is indeed the case for the extremal process of branching Brownian motion, and provides a characterization of the law of the clusters in terms of branching Brownian motions conditioned on performing "unusually high" jumps.

2.2. An auxiliary point process. Let $(\Omega', \mathcal{F}', P)$ be a probability space, and $Z : \Omega' \rightarrow \mathbb{R}_+$ with distribution as that of the limiting derivative martingale (1.6). (Expectation with respect to P will be denoted by E). Let $(\eta_i; i \in \mathbb{N})$ be the atoms of a Poisson point process on $(-\infty, 0)$ with density

$$\sqrt{\frac{2}{\pi}} \left(-x e^{-\sqrt{2}x}\right) dx. \quad (2.5)$$

For each $i \in \mathbb{N}$, consider independent branching Brownian motions with drift $-\sqrt{2}$, i.e. $\{x_k^{(i)}(t) - \sqrt{2}t; k \leq n_i(t)\}$, issued on $(\Omega', \mathcal{F}', P)$. Remark that, by (1.3) and (1.4), to given $i \in \mathbb{N}$,

$$\max_{k \leq n_i(t)} x_k^{(i)}(t) - \sqrt{2}t \rightarrow -\infty, \quad P\text{-a.s.} \quad (2.6)$$

The auxiliary point process of interest is the superposition of the iid BBM's with drift and shifted by $\eta_i + \frac{1}{\sqrt{2}} \log Z$:

$$\Pi_t \equiv \sum_{i,k} \delta_{\frac{1}{\sqrt{2}} \log Z + \eta_i + x_k^{(i)}(t) - \sqrt{2}t} \quad (2.7)$$

The existence and non-triviality of the process in the limit $t \rightarrow \infty$ is not straightforward, especially in view of (2.6). It will be proved by recasting the problem into the frame of convergence of solutions of the F-KPP equations to travelling waves, as in the proof of Theorem 2.2. It turns out that the density of the Poisson process points growing faster than exponentially as $x \rightarrow -\infty$ compensates for the fact that BBM's with drift wander off to $-\infty$.

Theorem 2.5 (The auxiliary point process). *Let \mathcal{E}_t be the extremal process (1.9) of BBM. Then*

$$\lim_{t \rightarrow \infty} \mathcal{E}_t \stackrel{\text{law}}{=} \lim_{t \rightarrow \infty} \Pi_t.$$

The above will follow from the fact that the Laplace functionals of $\lim_{t \rightarrow \infty} \Pi_t$ admits a representation of the form (2.4), and that the constants $C(\phi)$ in fact correspond.

Remark 2.6. *An elementary consequence of the above identification is that the extremal process \mathcal{E} shifted back by $\frac{1}{\sqrt{2}} \log Z$ is an infinitely divisible point process. The reader is referred to [25] for definitions and properties of such processes..*

We conjectured Theorem 2.5 in a recent paper [4], where it is pointed out that such a representation is a natural consequence of the results on the genealogies and the paths of the extremal particles in [3]. The proof of Theorem 2.5 provided here does not rely on such techniques. It is based on the analysis of Bramson [13] and the subsequent works of Chauvin-Rouault [15, 16], and Lalley and Sellke [27]. However, as discussed in Section 2.3, the results on the genealogies of [3] provides a useful heuristics. It is likely that such path techniques be an alternative approach to the Feynman-Kac representation on which the results of [13] (and thus those of our present paper) are based.

We now list some of the properties of the Poisson cluster process in the limit of large times (which by Theorem 2.5 coincide with those of the extremal process of BBM).

Proposition 2.7 (Poissonian nature of the cluster-extrema). *Consider Π_t^{ext} the point process obtained by retaining from Π_t the maximal particles of the BBM's,*

$$\Pi_t^{ext} \equiv \sum_i \delta_{\frac{1}{\sqrt{2}} \log Z + \eta_i + \max_k \{x_k^{(i)}(t) - \sqrt{2}t\}} \ .$$

Then $\lim_{t \rightarrow \infty} \Pi_t^{ext} \stackrel{law}{=} \text{PPP} \left(Z \sqrt{2} C e^{-\sqrt{2}x} dx \right)$, where C is the same constant appearing in (1.7). In particular, the maximum of $\lim_{t \rightarrow \infty} \Pi_t^{ext}$ has the same law as the limit law of the maximum of BBM.

The fact that the laws of the maximum of the cluster-extrema and of BBM correspond is a consequence of (1.7) and the formula for the maximum of a Poisson process.

The Poissonian nature of the cluster-extrema in the case of BBM was first proved in [4]. Given the equivalence of the extremal and cluster processes, the above thus comes as no surprise. We will provide a different proof from the one given in [4], which will be useful for the proof of Theorem 2.5.

The last ingredient of the proof of Theorem 2.1 is a characterization of the law of the clusters, that is the distribution of the points that are extremal and that come from the same atom of the Poisson process η . To this aim, it is necessary to understand which atoms in fact contribute to the extremal process. The following result is a good control of the location of such atoms that implies that the branching Brownian motion forming the clusters must perform unusually high jumps, of the order of $\sqrt{2}t + a\sqrt{t}$ for some $a > 0$.

Proposition 2.8. *Let $y \in \mathbb{R}$ and $\varepsilon > 0$ be given. There exists $0 < C_1 < C_2 < \infty$ and t_0 depending only on y and ε , such that*

$$\sup_{t \geq t_0} P \left[\exists_{i,k} : \eta_i + x_k^{(i)}(t) - \sqrt{2}t \geq y, \text{ and } \eta_i \notin [-C_1\sqrt{t}, -C_2\sqrt{t}] \right] < \varepsilon.$$

It will be shown that the conditional law of a BBM given the event that the maximum makes a displacement greater than $\sqrt{2}t$ exists in the limit $t \rightarrow \infty$ and, perhaps somewhat

surprisingly, does not depend on the displacement. This will entail that, seen from the cluster-extrema, the laws of the clusters are identical.

Proposition 2.9. *Let $x \equiv -a\sqrt{t} + b$ for some $a > 0, b \in \mathbb{R}$. In the limit $t \rightarrow \infty$, the conditional law of $\sum_{k \leq n(t)} \delta_{x+x_k(t)-\sqrt{2}t}$, given the event $\{x + \max_k x_k(t) - \sqrt{2}t > 0\}$, exists and does not depend on x . Moreover, $x + \max_k x_k(t) - \sqrt{2}t$ conditionally on the event $\{x + \max_k x_k(t) - \sqrt{2}t > 0\}$ converges weakly to an exponential random variable.*

2.3. A picture behind the theorem. Our understanding of BBM stems from results on the genealogies of extremal particles that were obtained in [3]. BBM can be seen as a Gaussian field of correlated random variables on the configuration space $\Sigma_t \equiv \{1, \dots, n(t)\}$. Conditionally upon a realization of the branching, the correlations among particles are given by the *genealogical distance*

$$Q_{ij}(t) \equiv \sup\{s \leq t : x_i(s) = x_j(s)\}, \quad i, j \in \Sigma_t. \quad (2.8)$$

The genealogical distance encodes all information on correlations, and in particular among extremal particles. As a first step towards a characterization of the correlation structure at the edge, we derived a characterization of the *paths* of extremal particles by identifying a mechanism of "entropic repulsion", which we shall recall.

Let $\alpha \in (0, 1/2)$. The *entropic envelope* is the curve

$$E_{\alpha,t}(s) \equiv \frac{s}{t}m(t) - e_{\alpha,t}(s), \quad (2.9)$$

with

$$e_{\alpha,t}(s) \equiv \begin{cases} s^\alpha, & 0 \leq s \leq t/2, \\ (t-s)^\alpha, & t/2 \leq s \leq t. \end{cases} \quad (2.10)$$

For $D \subset \mathbb{R}$, we denote by $\Sigma_t(D) \equiv \{k \in \Sigma_t : x_k(t) - m(t) \in D\}$ the set of particles falling at time t into the subset $D + m(t)$. A first result reads:

Theorem 2.10 ([3]). *For compact $D \subset \mathbb{R}$,*

$$\lim_{r_d, r_g \uparrow \infty} \lim_{t \uparrow \infty} \mathbb{P}[\exists k \in \Sigma_t(D) \text{ such that } x_k(s) \geq E_{\alpha,t}(s) \text{ for some } s \in (r_d, t - r_g)] = 0.$$

The content of the above is that extremal particles lie typically well below the maximal displacement, namely during the interval $(r_d, t - r_g)$. Remark that $r_d, r_g = o(t)$ as $t \rightarrow \infty$.

Proposition 2.10 allows to considerably restrict the correlations of particles at the edge.

Theorem 2.11 ([3]). *For compact $D \subset \mathbb{R}$,*

$$\lim_{r_d, r_g \uparrow \infty} \lim_{t \uparrow \infty} \mathbb{P}[\exists j, k \in \Sigma_t(D) \text{ such that } Q_{jk}(t) \in (r_d, t - r_g)] = 0.$$

According to this theorem, extremal particles can be split into groups of "weakly dependent" random variables (those particles with most recent common ancestors in $[0, r_d]$), or in groups of "heavily dependent" random variables (those particles with most recent common ancestors in $[t - r_g, t]$). The image which emerges from Theorems 2.10 and 2.11 is depicted in Figure 1 below.

Hence, branching happens at the very beginning, after which particles continue along *independent* paths, and start branching again only towards the end. It is not difficult to see [27, 4] that the branching at the beginning is responsible for the appearance of the

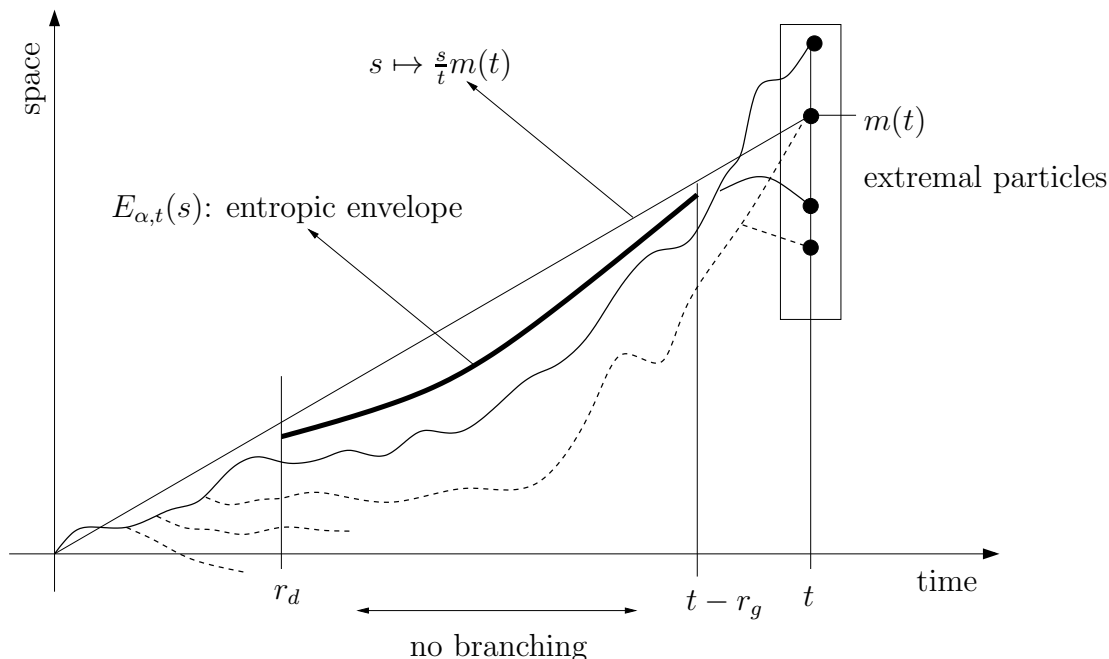


FIGURE 1. Genealogies of extremal particles

derivative martingale in the large time limit. On the other hand, the branching towards the end creates the clusters. By Theorem 2.10 these are branching Brownian motions of length r_g which have to perform displacements of order (at least) $\sqrt{2}r_g$; in fact, the displacements must be even bigger, in order to compensate for the "low" heights of the ancestors at time $t - r_g$. According to the picture, and conditionally on what happened up to time r_d (the initial branching) particles at time $t - r_g$ which have offsprings in the lead at the future time t have different ancestors at time r_d . Thus, one may expect that the point process of such ancestors should be ("close to") a *Poisson point process*; as such, only its density should matter. It is not difficult to see that the density of particles at time $t - r_g$ whose paths remain below the entropic envelope during all of the interval $(r_d, t - r_g)$ is in first approximation (and conditionally on the initial branching) of the form $-xe^{-\sqrt{2}x}dx$, for x large on the negative, in agreement with Theorem 2.5. This heuristics stands behind the existence of the Poisson cluster process representation.

The phenomenon described above is however more delicate than it might appear at first reading. This is in particular due to the following intriguing fact. Namely, the feature whether a particle alive at time $t - r_g$ has an offspring which makes it to the lead at the future time t is evidently not measurable w.r.t. the σ -field generated up to time $t - r_g$. In other words: *at any given time, leaders are offsprings of ancestors which are chosen according to whether their offsprings are leaders at the considered time!* We shall provide below yet another point of view on the issue which again suggests that such an intricate random thinning based on the future evolution eventually leads to the existence of the Poisson cluster process representation given in Theorem 2.5.

As it turns out, the Poisson cluster representation is of relevance for the study of spin glasses [10, 33], in particular within the frame of competing particle systems [CPS], a

wide-ranging set of ideas which address the statistical properties of extremal processes, as pursued by Aizenman and co-authors [1, 2, 31]. The CPS-approach can be seen as an attempt to formalize the so-called *cavity method* developed by Parisi and co-authors [30] for the study of spin glasses. The connection with Theorem 2.5 is as follows.

Let $\mathcal{E} = \sum_{i \in \mathbb{N}} \delta_{e_i}$ be the limiting extremal process of a BBM starting at zero. It is clear since $m(t) = m(t-s) + \sqrt{2}s + o(1)$ that the law of \mathcal{E} satisfies the following invariance property. For any $s \geq 0$,

$$\mathcal{E} \stackrel{\text{law}}{=} \sum_{i,k} \delta_{e_i + x_k^{(i)}(s) - \sqrt{2}s} \quad (2.11)$$

where $\{x_k^{(i)}(s); k \leq n^{(i)}(s)\}_{i \in \mathbb{N}}$ are iid BBM's. On the other hand, by Theorem 2.5,

$$\mathcal{E} \stackrel{\text{law}}{=} \lim_{s \rightarrow \infty} \sum_{i,k} \delta_{\eta_i + x_k^{(i)}(s) - \sqrt{2}s}, \quad (2.12)$$

where now $(\eta_i; i \in \mathbb{N})$ are the atoms of a PPP $\left(\sqrt{\frac{2}{\pi}} - xe^{-\sqrt{2}x}dx\right)$ shifted by $\frac{1}{\sqrt{2}} \log Z$.

The main idea of the CPS-approach is to characterize extremal processes as invariant measures under suitable, model-dependent stochastic mappings. In the case of branching Brownian motion, the mapping consists of adding to each ancestors e_i independent BBMs with drift $-\sqrt{2}$. This procedure randomly picks of the original ancestors only those with offsprings in the lead at some future time. But the random thinning is performed through independent random variables: this suggests that one may indeed replace the process of ancestors $\{e_i\}$ by a Poissonian process with suitable density.

Behind the random thinning, a crucial phenomenon of "energy vs. entropy" is at work. Under the light of (2.6), the probability that any such BBM with drift $-\sqrt{2}$ attached to the process of ancestors does not wander off to $-\infty$ vanishes in the limit of large times. On the other hand, the higher the position of the ancestors, the fewer one finds. A delicate balance must therefore be met, and only ancestors lying on a precise level below the lead can survive the random thinning. This is indeed the content of Proposition 2.8, and a fundamental ingredient in the CPS-heuristics: particles at the edge come from the tail in the past. A key step in identifying the equilibrium measure is thus to identify the tail. In the example of BBM, we are able to show, perhaps indirectly, that a good approximation of the tail is a Poisson process with density $-xe^{-\sqrt{2}x}dx$ (up to constant).

To make the above heuristics rigorous we rely on the analytic approach pioneered by Bramson, which highlights once more the power of the connection between BBM and the F-KPP equations. It may still be interesting to have a purely probabilistic proof, exploiting the detailed analysis of the path-properties of extremals particles. This represents, however, major technical challenges.

3. PROOFS

In what follows, $\{x_k(t), k \leq n(t)\}$ will always denote a branching Brownian motion of length t started in zero.

3.1. Technical tools. We start by stating two fundamental results that will be used extensively. First, McKean's insightful observation:

Lemma 3.1 ([29]). *Let $f : \mathbb{R} \rightarrow [0, 1]$ and $\{x_k(t) : k \leq n(t)\}$ a branching Brownian motion starting at 0. The function*

$$u(t, x) = \mathbb{E} \left[\prod_{k=1}^{n(t)} f(x + x_k(t)) \right]$$

is solution of the F-KPP equation (1.2) with $u(0, x) = f(x)$.

Second, the fundamental result by Bramson on the convergence of solutions of the F-KPP equation to travelling waves:

Theorem 3.2 (Theorem A [13]). *Let u be solution of the F-KPP equation (1.2) with $0 \leq u(0, x) \leq 1$. Then*

$$u(t, x + m(t)) \rightarrow \omega(x), \quad \text{uniformly in } x \text{ as } t \rightarrow \infty, \quad (3.1)$$

where ω is the unique solution (up to translation) of

$$\frac{1}{2}w'' + \sqrt{2}\omega' + \omega^2 - \omega = 0,$$

if and only if

1. *for some $h > 0$, $\limsup_{t \rightarrow \infty} \frac{1}{t} \log \int_t^{t(1+h)} (1 - u(0, y)) dy \leq -\sqrt{2}$;*
2. *and for some $\nu > 0$, $M > 0$, $N > 0$, $\int_x^{x+N} (1 - u(0, y)) dy > \nu$ for all $x \leq -M$.*

Moreover, if $\lim_{x \rightarrow \infty} e^{bx}(1 - u(0, x)) = 0$ for some $b > \sqrt{2}$, then one may choose

$$m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t. \quad (3.2)$$

It is to be noted that the necessary and sufficient conditions hold for uniform convergence in x . Pointwise convergence could hold when, for example, condition 2 is not satisfied. This is the case in Theorem 2.2.

It will be often convenient to consider the F-KPP equation in the following form:

$$u_t = \frac{1}{2}u_{xx} + u - \sum_{k=1}^{\infty} p_k u^k. \quad (3.3)$$

The solutions to (1.2) and (3.3) are simply related via the transformation $u \rightarrow 1 - u$. The following Proposition provides sharp approximations to the solutions of such equations.

Proposition 3.3. *Let u be a solution to the F-KPP equation (3.3) with initial data satisfying*

$$\int_0^{\infty} y e^{\sqrt{2}y} u(0, y) dy < \infty,$$

and such that $u(t, \cdot + m(t))$ converges. Define

$$\psi(r, t, X + \sqrt{2}t) \equiv \frac{e^{-\sqrt{2}X}}{\sqrt{2\pi(t-r)}} \int_0^{\infty} u(r, y' + \sqrt{2}r) e^{y'\sqrt{2}} e^{-\frac{(y'-X)^2}{2(t-r)}} (1 - e^{-2y' \frac{(X + \frac{3}{2\sqrt{2}} \log t)}{t-r}}) dy' \quad (3.4)$$

Then for r large enough, $t \geq 8r$, and $X \geq 8r - \frac{3}{2\sqrt{2}} \log t$,

$$\gamma^{-1}(r)\psi(r, t, X + \sqrt{2t}) \leq u(t, X + \sqrt{2t}) \leq \gamma(r)\psi(r, t, X + \sqrt{2t}), \quad (3.5)$$

for some $\gamma(r) \downarrow 1$ as $r \rightarrow \infty$.

With the notations from the above Proposition, the function ψ thus fully captures the large space-time behavior of the solution to the F-KPP equations. We will make extensive use of (3.5), mostly when both X and t are large in the positive, in which case the dependence on X becomes particularly easy to handle.

Proof of Proposition 3.3. For $T > 0$ and $0 < \alpha < \beta < \infty$, let $\{\mathfrak{z}_{\alpha, \beta}^T(s), 0 \leq s \leq T\}$ denote a Brownian bridge of length T starting in α and ending in β .

It has been proved by Bramson (see [13, Proposition 8.3]) that for u satisfying the assumptions in the Proposition 3.3, the following holds:

- (1) for r large enough, $t \geq 8r$ and $x \geq m(t) + 8r$

$$u(t, x) \geq C_1(r)e^{t-r} \int_{-\infty}^{\infty} u(r, y) \frac{e^{-\frac{(x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}} \mathbb{P}[\mathfrak{z}_{x,y}^{t-r}(s) > \overline{\mathcal{M}}_{r,t}^x(t-s), s \in [0, t-r]] dy$$

and

$$u(t, x) \leq C_2(r)e^{t-r} \int_{-\infty}^{\infty} u(r, y) \frac{e^{-\frac{(x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}} \mathbb{P}[\mathfrak{z}_{x,y}^{t-r}(s) > \underline{\mathcal{M}}'_{r,t}(t-s), s \in [0, t-r]] dy$$

where the functions $\overline{\mathcal{M}}_{r,t}^x(t-s)$, $\underline{\mathcal{M}}'_{r,t}(t-s)$ satisfy

$$\underline{\mathcal{M}}'_{r,t}(t-s) \leq n_{r,t}(t-s) \leq \overline{\mathcal{M}}_{r,t}^x(t-s),$$

for $n_r(s)$ being the linear interpolation between $\sqrt{2}r$ at time r and $m(t)$ at time t . Moreover, $C_1(r) \uparrow 1$, $C_2(r) \downarrow 1$ as $r \rightarrow \infty$.

- (2) If $\psi_1(r, t, x)$ and $\psi_2(r, t, x)$ denote respectively the lower and upper bound to $u(t, x)$, we have

$$1 \leq \frac{\psi_2(r, t, x)}{\psi_1(r, t, x)} \leq \gamma(r)$$

where $\gamma(r) \downarrow 1$ as $r \rightarrow \infty$.

Hence, if we denote by

$$\widehat{\psi}(r, t, x) = e^{t-r} \int_{-\infty}^{\infty} u(r, y) \frac{e^{-\frac{(x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}} \mathbb{P}(\mathfrak{z}_{x,y}^{t-r}(s) > n_{r,t}(t-s), s \in [0, t-r]) dy,$$

we have by domination $\psi_1 \leq \widehat{\psi} \leq \psi_2$. Therefore, for r, t and x large enough

$$\frac{u(t, x)}{\widehat{\psi}(r, t, x)} \leq \frac{\psi_2(r, t, x)}{\widehat{\psi}(r, t, x)} \leq \frac{\psi_2(r, t, x)}{\psi_1(r, t, x)} \leq \gamma(r), \quad (3.6)$$

and

$$\frac{u(t, x)}{\widehat{\psi}(r, t, x)} \geq \frac{1}{\gamma(r)}. \quad (3.7)$$

Combining (3.6) and (3.7) we thus get

$$\gamma^{-1}(r)\widehat{\psi}(r, t, x) \leq u(t, x) \leq \gamma(r)\widehat{\psi}(r, t, x). \quad (3.8)$$

We now consider $X \geq 8r - \frac{3}{2\sqrt{2}} \log t$, and obtain from (3.8) that

$$\gamma^{-1}(r)\widehat{\psi}(r, t, X + \sqrt{2}t) \leq u(t, X + \sqrt{2}t) \leq \gamma(r)\widehat{\psi}(r, t, X + \sqrt{2}t). \quad (3.9)$$

The probability involving the Brownian bridge in the definition of $\widehat{\psi}$ can be explicitly computed. The probability of a Brownian bridge of length t to remain below the interpolation of $A > 0$ at time 0 and $B > 0$ at time t is $1 - e^{-2AB/t}$, see e.g. [32]. In the above setting the length is $t - r$, $A = \sqrt{2}t + x - m(t) = x + \frac{3}{2\sqrt{2}} \log t > 0$ for t large enough and $B = y - \sqrt{2}r = y'$. Using this, together with the fact that $\mathbb{P}(\mathfrak{Z}_{x,y}^{t-r}(s) > n_{r,t}(t-s), s \in [0, t-r])$ is 0 for $B = y' < 0$, and by change of variable $y' = y + \sqrt{2}t$ in the integral appearing in the definition of $\widehat{\psi}$, we get

$$\begin{aligned} \widehat{\psi}(r, t, X + \sqrt{2}t) &= \frac{e^{-\sqrt{2}X}}{\sqrt{2\pi(t-r)}} \int_0^\infty u(r, y' + \sqrt{2}r) e^{y'\sqrt{2}} e^{-\frac{(y'-X)^2}{2(t-r)}} (1 - e^{-2y' \frac{(X + \frac{3}{2\sqrt{2}} \log t)}{t-r}}) dy' \\ &= \psi(r, t, X + \sqrt{2}t). \end{aligned} \quad (3.10)$$

This, together with (3.8), concludes the proof of the proposition. \square

The bounds in (3.5) have been used by Chauvin and Rouault to compute the probability of deviations of the maximum of BBM, see Lemma 2 [15]. Their reasoning applies to solutions of the F-KPP equation with other initial conditions than those corresponding to the maximum. We give the statement below, and reproduce Chauvin and Rouault's proof in a general setting for completeness.

Proposition 3.4. *Let the assumptions of Proposition 3.3 be satisfied, and assume furthermore that $y_0 = \sup\{y : u(0, y) > 0\}$ is finite. Then,*

$$\lim_{t \rightarrow \infty} e^{x\sqrt{2}} \frac{t^{3/2}}{\log t} \psi(r, t, x + \sqrt{2}t) = \frac{3}{2\sqrt{\pi}} \int_0^\infty y e^{y\sqrt{2}} u(r, y + \sqrt{2}r) dy$$

Moreover, the limit of the right-hand side exists as $r \rightarrow \infty$, and it is positive and finite.

Proof. The first claim is straightforward if we can take the limit $t \rightarrow \infty$ inside the integral in the definition of ψ . We need to justify this using dominated convergence. Since $e^{-x} \geq 1 - x$ for $x > 0$, the integrand in the definition of ψ times $e^{x\sqrt{2}} \frac{t^{3/2}}{\log t}$ is smaller than

$$(cte) y' e^{y'\sqrt{2}} u(r, y' + \sqrt{2}r). \quad (3.11)$$

It remains to show that (3.11) is integrable in $y' \geq 0$. To see this, let $u^{(2)}(t, x)$ be the solution to $\partial_t u^{(2)} = \frac{1}{2} u_{xx}^{(2)} - u^{(2)}$ [the *linearised* F-KPP-equation (3.3)] with initial conditions $u^{(2)}(0, x) = u(0, x)$. By the maximum principle for nonlinear, parabolic pde's, see e.g. [13, Corollary 1, p.29],

$$u(t, x) \leq u^{(2)}(t, x) \quad (3.12)$$

Moreover, by the Feynman-Kac representation and the definition of y_0 ,

$$u^{(2)}(t, x) \leq e^t \int_{-\infty}^{y_0} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} dy, \quad (3.13)$$

and for any $x > 0$ we thus have the bound

$$u^{(2)}(t, x) \leq e^t e^{-\frac{x^2}{2t}} e^{\frac{y_0 x}{t}}. \quad (3.14)$$

Hence,

$$u(r, y + \sqrt{2}r) \leq e^{-\frac{y^2}{2r}} e^{\frac{y_0 y}{2r}} e^{-y\sqrt{2}}. \quad (3.15)$$

The upper bound is integrable over the desired measure since

$$\int_0^\infty y e^{\sqrt{2}y} e^{-\frac{y^2}{2r}} e^{\frac{y_0 y}{2r}} e^{-y\sqrt{2}} dy = \int_0^\infty y e^{-\frac{y^2}{2r}} e^{\frac{y_0 y}{2r}} dy < \infty. \quad (3.16)$$

Therefore dominated convergence can be applied and the first part of the Proposition follows.

It remains to show that

$$\lim_{r \rightarrow \infty} \int_0^\infty y e^{y\sqrt{2}} u(r, y + \sqrt{2}r) dy \text{ exists and is finite.}$$

Write $C(r)$ for the integral. By Proposition 3.3, for r large enough,

$$\limsup_{t \rightarrow \infty} e^{x\sqrt{2}} \frac{t^{3/2}}{\log t} u(t, x + \sqrt{2}t) \leq \gamma(r) \lim_{t \rightarrow \infty} e^{x\sqrt{2}} \frac{t^{3/2}}{\log t} \psi(r, t, x + \sqrt{2}t) = C(r) \gamma(r),$$

and

$$\liminf_{t \rightarrow \infty} e^{x\sqrt{2}} \frac{t^{3/2}}{\log t} u(t, x + \sqrt{2}t) \geq \gamma(r)^{-1} \lim_{t \rightarrow \infty} e^{x\sqrt{2}} \frac{t^{3/2}}{\log t} \psi(r, t, x + \sqrt{2}t) = C(r) \gamma(r)^{-1},$$

Therefore since $\gamma(r) \rightarrow 1$

$$\limsup_{t \rightarrow \infty} e^{x\sqrt{2}} \frac{t^{3/2}}{\log t} u(t, x + \sqrt{2}t) \leq \liminf_{r \rightarrow \infty} C(r)$$

and

$$\liminf_{t \rightarrow \infty} e^{x\sqrt{2}} \frac{t^{3/2}}{\log t} u(t, x + \sqrt{2}t) \geq \limsup_{r \rightarrow \infty} C(r).$$

It follows that $\lim_{r \rightarrow \infty} C(r) =: C$ exists and so does $\lim_{t \rightarrow \infty} e^{x\sqrt{2}} \frac{t^{3/2}}{\log t} u(t, x + \sqrt{2}t)$. Moreover $C > 0$ otherwise

$$\lim_{t \rightarrow \infty} e^{x\sqrt{2}} \frac{t^{3/2}}{\log t} u(t, x + \sqrt{2}t) = 0 \quad (3.17)$$

which is impossible since

$$\lim_{t \rightarrow \infty} e^{x\sqrt{2}} \frac{t^{3/2}}{\log t} u(t, x + \sqrt{2}t) \geq C(r)/\gamma(r)$$

for r large enough but finite ($\gamma(r)$ and $C(r)$ are finite for r finite). Moreover $C < \infty$, otherwise

$$\lim_{t \rightarrow \infty} e^{x\sqrt{2}} \frac{t^{3/2}}{\log t} u(t, x + \sqrt{2}t) = \infty,$$

which is impossible since $\lim_{t \rightarrow \infty} e^{x\sqrt{2} \frac{t^{3/2}}{\log t}} u(t, x + \sqrt{2t}) \leq C(r)\gamma(r)$ for r large enough, but finite. \square

3.2. Existence of a limiting process.

Proof of Proposition 2.2. It suffices to show that, for $\phi \in \mathcal{C}_c(\mathbb{R})$ positive, the Laplace transform $\Psi_t(\phi)$, defined in (2.3), of the extremal process of branching Brownian motion converges.

Remark first that this limit cannot be 0, since in the case of BBM it can be checked [3] that

$$\lim_{N \rightarrow \infty} \lim_{t \rightarrow \infty} \mathbb{P}[\mathcal{E}_t(B) > N] = 0, \text{ for any bounded measurable set } B \subset \mathbb{R},$$

hence the limiting point process must be locally finite.

For convenience, we define $\max \mathcal{E}_t \equiv \max_{k \leq n(t)} x_k(t) - m(t)$. By Theorem 3.2 applied to the function

$$u(t, \delta + m(t)) = \mathbb{E} \left[\prod_{k=1}^{n(t)} \mathbb{1}_{\{x_k(t) - m(t) \leq \delta\}} \right] = \mathbb{P}[\max \mathcal{E}_t \leq \delta]$$

it holds that

$$\lim_{\delta \rightarrow \infty} \lim_{t \rightarrow \infty} 1 - u(t, \delta + m(t)) = \lim_{\delta \rightarrow \infty} 1 - \omega(\delta) = 0. \quad (3.18)$$

Now consider for $\delta > 0$

$$\begin{aligned} \mathbb{E} \left[\exp \left(- \int \phi d\mathcal{E}_t \right) \right] &= \mathbb{E} \left[\exp \left(- \int \phi d\mathcal{E}_t \right) \mathbb{1}_{\{\max \mathcal{E}_t \leq \delta\}} \right] \\ &\quad + \mathbb{E} \left[\exp \left(- \int \phi d\mathcal{E}_t \right) \mathbb{1}_{\{\max \mathcal{E}_t > \delta\}} \right] \end{aligned} \quad (3.19)$$

Note that by (3.18), the second term on the r.h.s of (3.19) satisfies

$$\limsup_{\delta \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbb{E} \left[\exp \left(- \int \phi d\mathcal{E}_t \right) \mathbb{1}_{\{\max \mathcal{E}_t > \delta\}} \right] \leq \limsup_{\delta \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbb{P}[\max \mathcal{E}_t > \delta] = 0.$$

It remains to address the first term on the r.h.s of (3.19). Write for convenience

$$\Psi_t^\delta(\phi) \equiv \mathbb{E} \left[\exp \left(- \int \phi d\mathcal{E}_t \right) \mathbb{1}_{\{\max \mathcal{E}_t \leq \delta\}} \right].$$

We claim that the limit

$$\lim_{\delta \rightarrow \infty} \lim_{t \rightarrow \infty} \Psi_t^\delta(\phi) \equiv \Psi(\phi) \quad (3.20)$$

exists, and is strictly smaller than one. To see this, set

$$g_\delta(x) \equiv e^{-\phi(x)} \mathbb{1}_{\{x \leq \delta\}},$$

and

$$u_\delta(t, x) \equiv \mathbb{E} \left[\prod_{k \leq n(t)} g_\delta(-x + x_k(t)) \right]. \quad (3.21)$$

By Lemma 3.1, u_δ is then solution to the F-KPP equation with $u_\delta(0, x) = g_\delta(-x)$. Moreover, $g_\delta(-x) = 1$ for x large enough in the positive, and $g_\delta(-x) = 0$ for x large enough in the negative, so that conditions (1) and (2) of Theorem 3.2 are satisfied as well as (3.2)

on the form of $m(t)$. Note that this would not be the case without the presence of the cutoff. Therefore

$$u_\delta(t, x + m(t)) = \mathbb{E} \left[\prod_{k=1}^{n(t)} g_\delta(-x + x_k(t) - m(t)) \right] \quad (3.22)$$

converges as $t \rightarrow \infty$ uniformly in x by Theorem 3.2. But

$$\begin{aligned} \Psi_t^\delta(\phi) &= \mathbb{E} \left[\exp \left(- \int \phi d\mathcal{E}_t \right) \mathbb{1}_{\{\max \mathcal{E}_t \leq \delta\}} \right] \\ &= \mathbb{E} \left[\prod_{k \leq n(t)} \exp \left(- \phi(x_k(t) - m(t)) \right) \mathbb{1}_{\{x_k(t) - m(t) \leq \delta\}} \right] \\ &= \mathbb{E} \left[\prod_{k \leq n(t)} g_\delta(x_k(t) - m(t)) \right] = u_\delta(t, 0 + m(t)), \end{aligned} \quad (3.23)$$

and therefore the limit $\lim_{t \rightarrow \infty} \Psi_t^\delta(\phi) \equiv \Psi^\delta(\phi)$ exists. But the function $\delta \mapsto \Psi^\delta(\phi)$ is increasing and smaller than one, by construction. Therefore, $\lim_{\delta \rightarrow \infty} \Psi^\delta(\phi) = \Psi(\phi)$ exists. Moreover, nonnegativity of ϕ implies that $\Psi_t^\delta(\phi) \leq \mathbb{P}[\max \mathcal{E}_t \leq \delta]$: taking the limit $t \rightarrow \infty$ first and $\delta \rightarrow \infty$ next thus shows that $\Psi(\phi) < 1$, and concludes the proof. \square

3.3. The process of cluster-extrema. Proposition 3.4 can be used to obtain an elementary proof of the convergence of the process of the cluster-extrema towards the PPP $\left(CZ\sqrt{2}e^{-\sqrt{2}x}dx\right)$. We start by proving two lemmas that will be of use later on.

Lemma 3.5. *Let $u(t, x)$ be a solution to F-KPP with initial condition $u(0, x)$ satisfying the assumption of Proposition 3.4. Let*

$$C \equiv \lim_{r \rightarrow \infty} \sqrt{\frac{2}{\pi}} \int_0^\infty y e^{y\sqrt{2}} u(t, y + \sqrt{2}r) dy ,$$

then for any $x \in \mathbb{R}$:

$$\lim_{r \rightarrow \infty} \sqrt{\frac{2}{\pi}} \int_0^\infty y e^{y\sqrt{2}} u(t, x + y + \sqrt{2}t) = C e^{-\sqrt{2}x} . \quad (3.24)$$

Proof. By Proposition 3.4

$$\begin{aligned} \lim_{r \rightarrow \infty} \int_0^\infty y e^{y\sqrt{2}} u(t, x + y + \sqrt{2}t) &= e^{-x\sqrt{2}} \frac{2\sqrt{\pi}}{3} \lim_{t \rightarrow \infty} e^{x\sqrt{2}} \frac{t^{3/2}}{\log t} u(t, x + \sqrt{2}t) \\ &= e^{-x\sqrt{2}} \lim_{r \rightarrow \infty} \int_0^\infty y e^{y\sqrt{2}} u(t, y + \sqrt{2}r) \end{aligned} \quad (3.25)$$

\square

A straightforward consequence of the lemma, taking $u(0, x) = \mathbb{1}_{\{x > a\}}$, $a \in \mathbb{R}$, is the following vague convergence of the maximum when integrated over the appropriate density.

Lemma 3.6. *For any continuous function $h : \mathbb{R} \rightarrow \mathbb{R}$ that is bounded at $+\infty$ and vanishes at $-\infty$,*

$$\begin{aligned} \lim_{t \rightarrow \infty} \int_{-\infty}^0 \left\{ \int_{\mathbb{R}} h(x) \mathbb{P} \left(\max_i x_i(t) - \sqrt{2}t + y \in dx \right) \right\} \sqrt{\frac{2}{\pi}} (-ye^{-\sqrt{2}y}) dy \\ = \int_{\mathbb{R}} h(a) \sqrt{2} C e^{-\sqrt{2}a} da . \end{aligned}$$

Proof of Proposition 2.7. Consider

$$E \exp - \sum_i \phi(\eta_i + M^i(t) - \sqrt{2}t)$$

where $\eta = (\eta_i)$ is Poisson with density $\sqrt{\frac{2}{\pi}}(-xe^{-\sqrt{2}x})dx$ on $(-\infty, 0)$ and $M^i(t) \equiv \max_k x_k^{(i)}(t)$. We show that

$$\lim_{t \rightarrow \infty} E \exp - \sum_i \phi(\eta_i + M^i(t) - \sqrt{2}t) = \exp - C \int_{\mathbb{R}} (1 - e^{-\phi(x)}) e^{-\sqrt{2}x} dx . \quad (3.26)$$

Since the underlying process is Poisson and the M^i 's are iid,

$$E \exp - \sum_i \phi(\eta_i + M^i(t) - \sqrt{2}t) = \exp - \int_{-\infty}^0 \left(1 - E \left[e^{-\phi(x + M(t) - \sqrt{2}t)} \right] \right) \sqrt{\frac{2}{\pi}} (-xe^{-\sqrt{2}x}) dx .$$

The result then follows from Lemma 3.6 after taking the limit. \square

3.4. The Laplace functional and the F-KPP equation.

Proof of Proposition 2.3. The proof of the proposition will be broken into proving two lemmas.

In the first lemma we establish an integral representation for the Laplace functionals of the extremal process of BBM which are truncated by a certain cutoff; in the second lemma we show that the results continues to holds when the cutoff is lifted. Throughout this section, $\phi : \mathbb{R} \rightarrow [0, \infty)$ is a non-negative continuous function with compact support.

Lemma 3.7. *Consider*

$$u_\delta(t, x) \equiv 1 - \mathbb{E} \left[\exp \left(- \sum_k \phi(-x + x_k(t)) \right) 1_{\{\max_{k \leq n(t)} -x + x_k(t) \leq \delta\}} \right] .$$

Then $u_\delta(t, x)$ is the solution of the F-KPP equation (3.3) with initial condition $u_\delta(0, x) = 1 - e^{-\phi(-x)} 1_{(-x \leq \delta)}$. Moreover,

$$\lim_{t \rightarrow \infty} u_\delta(t, x + m(t)) = 1 - \mathbb{E} \left[\exp - C(\phi, \delta) Z e^{-\sqrt{2}x} \right] , \quad (3.27)$$

where

$$C(\phi, \delta) = \lim_{t \rightarrow \infty} \sqrt{\frac{2}{\pi}} \int_0^\infty u_\delta(t, y + \sqrt{2}t) y e^{\sqrt{2}y} dy . \quad (3.28)$$

Proof. The first part of the Lemma is proved in the proof of Theorem 2.2, whereas (3.27) follows from Theorem 3.2 and the representation (1.7). It remains to prove (3.28). The proof is a refinement of Proposition 3.4 that recovers the asymptotics (1.8).

For u_δ as above, let $\psi(r, t, x)$ be its approximation as in Proposition 3.3 and choose x, r so that $x \geq m(t) + 8r$. By Proposition 3.3 we then have the bounds

$$\frac{1}{\gamma(r)}\psi(r, t, x + m(t)) \leq u_\delta(t, x + m(t)) \leq \gamma(r)\psi(r, t, x + m(t)) \quad (3.29)$$

where

$$\psi(r, t, x + m(t)) = \frac{t^{3/2}e^{-\sqrt{2}x}}{\sqrt{2\pi}(t-r)} \int_0^\infty u_\delta(r, y' + \sqrt{2}r) e^{y'\sqrt{2}} e^{-\frac{(y'-x+\frac{3}{2\sqrt{2}\log t})^2}{2(t-r)}} (1 - e^{-2\frac{y'x}{t-r}}) dy'.$$

Using dominated convergence as in Proposition 3.4, one gets

$$\lim_{t \rightarrow \infty} \psi(r, t, x + m(t)) = \frac{2xe^{-\sqrt{2}x}}{\sqrt{2\pi}} \int_0^\infty u_\delta(r, y' + \sqrt{2}r) y' e^{y'\sqrt{2}} dy'.$$

Putting this back in (3.29),

$$\frac{1}{\gamma(r)}C(r) \leq \lim_{t \rightarrow \infty} \frac{u_\delta(t, x + m(t))}{xe^{-\sqrt{2}x}} \leq \gamma(r)C(r), \quad (3.30)$$

for $C(r) \equiv \sqrt{\frac{2}{\pi}} \int_0^\infty u_\delta(r, y' + \sqrt{2}r) y' e^{y'\sqrt{2}} dy$, and $x > 8r$. We know that $\lim_{r \rightarrow \infty} C(r) \equiv C > 0$ exists by Proposition 3.4. Thus taking $x = 9r$, letting $r \rightarrow \infty$ in (3.30), and using that $\gamma(r) \downarrow 1$, one has

$$\lim_{x \rightarrow \infty} \lim_{t \rightarrow \infty} \frac{u_\delta(t, x + m(t))}{xe^{-\sqrt{2}x}} = \lim_{r \rightarrow \infty} \sqrt{\frac{2}{\pi}} \int_0^\infty u_\delta(r, y' + \sqrt{2}r) y' e^{y'\sqrt{2}} dy.$$

On the other hand, the representation (1.7) and the asymptotics (1.8) yield

$$\lim_{x \rightarrow \infty} \lim_{t \rightarrow \infty} \frac{u_\delta(t, x + m(t))}{xe^{-\sqrt{2}x}} = \frac{1 - \mathbb{E} \left[\exp \left(-C(\phi, \delta) Z e^{-\sqrt{2}x} \right) \right]}{xe^{-\sqrt{2}x}} = C(\phi, \delta).$$

The claim follows from the last two equations. \square

The results of Lemma 3.7 also hold when the cutoff is removed. The proof shows (non-uniform) convergence of the solution of the F-KPP equation when one condition of Theorem 3.2 is not fulfilled. With an appropriate continuity argument, a Lalley-Sellke type representation is recovered.

Lemma 3.8. *Let $u(t, x), u_\delta(t, x)$ be solutions of the F-KPP equation (3.3) with initial condition $u(0, x) = 1 - e^{-\phi(-x)}$ and $u_\delta(t, x) = 1 - e^{-\phi(-x)} \mathbb{1}_{\{-x \leq \delta\}}$, respectively. Set $C(\delta, \phi) \equiv \lim_{t \rightarrow \infty} \sqrt{\frac{2}{\pi}} \int_0^\infty u_\delta(t, y + \sqrt{2}t) y e^{\sqrt{2}y} dy$. Then*

$$\lim_{t \rightarrow \infty} u(t, x + m(t)) = 1 - \mathbb{E} \left[\exp -C(\phi) Z e^{-\sqrt{2}x} \right],$$

with

$$C(\phi) \equiv \lim_{t \rightarrow \infty} \sqrt{\frac{2}{\pi}} \int_0^\infty u(t, y + \sqrt{2}t) y e^{\sqrt{2}y} dy = \lim_{\delta \rightarrow \infty} C(\phi, \delta).$$

Proof. It is straightforward to check that

$$0 \leq u_\delta(t, x) - u(t, x) \leq \mathbb{P}(\max x_k(t) > \delta + x), \quad (3.31)$$

from which it follows that

$$\begin{aligned} & \int_0^\infty u_\delta(t, x + \sqrt{2}t) x e^{\sqrt{2}x} dx - \int_0^\infty \mathbb{P} \left[\max x_k(t) - \sqrt{2}t > \delta + x \right] x e^{\sqrt{2}x} dx \\ & \leq \int_0^\infty u(t, x + \sqrt{2}t) x e^{\sqrt{2}x} dx \\ & \leq \int_0^\infty u_\delta(t, x + \sqrt{2}t) x e^{\sqrt{2}x} dx. \end{aligned} \quad (3.32)$$

Define

$$\begin{aligned} F(t, \delta) & \equiv \int_0^\infty u_\delta(t, x + \sqrt{2}t) x e^{\sqrt{2}x} dx, \\ U(t) & \equiv \int_0^\infty u(t, x + \sqrt{2}t) x e^{\sqrt{2}x} dx, \end{aligned}$$

and

$$M(t, \delta) \equiv \int_0^\infty \mathbb{P} \left[\max x_k(t) - \sqrt{2}t > \delta + x \right] x e^{\sqrt{2}x} dx.$$

The inequalities in (3.32) then read

$$F(t, \delta) - M(t, \delta) \leq F(t) \leq F(t, \delta). \quad (3.33)$$

We claim that

$$\lim_{\delta \rightarrow \infty} \lim_{t \rightarrow \infty} M(t, \delta) = 0. \quad (3.34)$$

We postpone the proof of this, and remark that Proposition 3.4 implies that to given δ , $\lim_{t \rightarrow \infty} F(t, \delta) \equiv F(\delta)$ exists and is strictly positive. We thus deduce from (3.33) that

$$\liminf_{\delta \rightarrow \infty} F(\delta) \leq \liminf_{t \rightarrow \infty} U(t) \leq \limsup_{t \rightarrow \infty} U(t) \leq \limsup_{\delta \rightarrow \infty} F(\delta). \quad (3.35)$$

We claim that $\lim_{\delta \rightarrow \infty} F(\delta)$ exists, is strictly positive and finite. To see this, we first observe that the function $\delta \rightarrow F(\delta)$ is by construction decreasing, and positive, therefore the limit $\lim_{\delta \rightarrow \infty} F(\delta)$ exists. Strict positivity is proved in a somewhat indirect fashion: we proceed by contradiction, and rely on the convergence of the process of cluster extrema. Assume that

$$\lim_{\delta \rightarrow \infty} F(\delta) = 0, \quad (3.36)$$

and thus that

$$\lim_{t \rightarrow \infty} U(t) = 0. \quad (3.37)$$

Using the form of the Laplace functional of a Poisson process, we have that

$$\begin{aligned}
& E \left[\exp \left(- \int \phi(x) \Pi_t(dx) \right) \right] = \\
& = E \exp \left[-Z \sqrt{\frac{2}{\pi}} \int_{-\infty}^0 \left(1 - \mathbb{E} \left[\exp - \sum_{k \leq n(t)} \phi(x + x_k(t) - \sqrt{2}t) \right] \right) \left\{ -x e^{-\sqrt{2}x} \right\} dx \right] \\
& \stackrel{x \rightarrow -x}{=} E \exp \left[-Z \sqrt{\frac{2}{\pi}} \int_0^{\infty} \left(1 - \mathbb{E} \left[\exp - \sum_{k \leq n(t)} \phi(-x + x_k(t) - \sqrt{2}t) \right] \right) x e^{\sqrt{2}x} dx \right] \\
& = E \left[\exp \left(-Z \sqrt{\frac{2}{\pi}} U(t) \right) \right].
\end{aligned} \tag{3.38}$$

Therefore, (3.37) would imply that

$$\lim_{t \rightarrow \infty} E \left[\exp \left(- \int \phi(x) \Pi_t(dx) \right) \right] = E \left[\exp \left(-Z \sqrt{\frac{2}{\pi}} \lim_{t \rightarrow \infty} U(t) \right) \right] = 1. \tag{3.39}$$

This cannot hold. In fact, for Π_t^{ext} the process of the cluster-extrema defined earlier, one has the obvious bound

$$E \left[\exp \left(- \int \phi(x) \Pi_t(dx) \right) \right] \leq E \left[\exp \left(- \int \phi(x) \Pi_t^{\text{ext}}(dx) \right) \right]. \tag{3.40}$$

Since the process of cluster-extrema converges, by Proposition 2.7, to a PPP $(CZ e^{-\sqrt{2}x} dx)$,

$$\lim_{t \rightarrow \infty} E \left[\exp \left(- \int \phi(x) \Pi_t(dx) \right) \right] \leq E \left[\exp \left(-CZ \int \{1 - e^{-\phi(x)}\} e^{-\sqrt{2}x} dx \right) \right] < 1. \tag{3.41}$$

This contradicts (3.39) and therefore also (3.37).

It remains to prove (3.34). For this, we shall use an upper bound for the right tail of the law of the maximum of BBM established in [4]. It is a consequence of tight bounds established by Bramson [13, Prop. 8.2].

Lemma 3.9. [4, Cor. 10] *For $X > 1$, and $t \geq t_o$ (for t_o a numerical constant),*

$$\mathbb{P} \left[\max_{k \leq n(t)} x_k(t) - m(t) \geq X \right] \leq \rho \cdot X \cdot \exp \left(-\sqrt{2}X - \frac{X^2}{2t} + \frac{3}{2\sqrt{2}} X \frac{\log t}{t} \right). \tag{3.42}$$

for some constant $\rho > 0$.

Now since $\sqrt{2}t = m(t) + \frac{3}{2\sqrt{2}} \log t$,

$$\begin{aligned}
& \int_0^\infty \mathbb{P} \left[\max x_k(t) - \sqrt{2}t > \delta + x \right] x e^{\sqrt{2}x} dx \\
&= \int_0^\infty \mathbb{P} \left[\max x_k(t) - m(t) > \delta + x + \frac{3}{2\sqrt{2}} \log t \right] x e^{\sqrt{2}x} dx \\
&= \frac{e^{-\sqrt{2}\delta}}{t^{3/2}} \int_{\delta + \frac{3}{2\sqrt{2}} \log t}^\infty \mathbb{P} [\max x_k(t) - m(t) > y] y e^{\sqrt{2}y} dy \\
&\quad - \left(\delta + \frac{3}{2\sqrt{2}} \log t \right) \frac{e^{-\sqrt{2}\delta}}{t^{3/2}} \int_{\delta + \frac{3}{2\sqrt{2}} \log t}^\infty \mathbb{P} [\max x_k(t) - m(t) > y] e^{\sqrt{2}y} dy,
\end{aligned} \tag{3.43}$$

the last line by change of variable. We address the large time limit of the first term on the right-hand side above. The second term is handled similarly. By Lemma 3.9, the first term is bounded, up to constant, by

$$\begin{aligned}
& \frac{e^{-\sqrt{2}\delta}}{t^{3/2}} \int_{\delta + \frac{3}{2\sqrt{2}} \log t}^\infty y^2 \exp \left(-\frac{y^2}{2t} + \frac{3}{2\sqrt{2}} y \frac{\log t}{t} \right) dy \\
&= \frac{e^{-\sqrt{2}\delta}}{t^{3/2}} \exp \left[\frac{9}{16} \frac{(\log t)^2}{t} \right] \int_{\delta + \frac{3}{2\sqrt{2}} \log t}^\infty y^2 e^{-\frac{(y - 3 \log t / (2\sqrt{2}))^2}{2t}} dy \\
&\leq 2 \cdot \frac{e^{-\sqrt{2}\delta}}{t^{3/2}} \int_\delta^\infty \left(z + \frac{3}{2\sqrt{2}} \log t \right)^2 e^{-z^2/2t} dz,
\end{aligned} \tag{3.44}$$

where in the last line we used that $\exp \left[\frac{9}{16} \frac{(\log t)^2}{t} \right] = 1 + o(1) \leq 2$, as $t \rightarrow \infty$. By developing the square $\left(z + \frac{3}{2\sqrt{2}} \log t \right)^2$ one easily sees that the only contribution which is not vanishing in the limit comes from the z^2 -term, for which we have

$$2 \cdot \frac{e^{-\sqrt{2}\delta}}{t^{3/2}} \cdot \int_\delta^\infty z^2 e^{-z^2/2t} dz \leq \rho \cdot e^{-\sqrt{2}\delta} \rightarrow 0, \tag{3.45}$$

as $\delta \rightarrow \infty$. This implies (3.34) and concludes the proof of Lemma 3.8. \square

Combining the assertions of Lemma 3.7 and Lemma 3.8 yields the assertion of Proposition 2.3. \square

Proposition 3.8 yields a short proof of Theorem 2.5.

Proof of Theorem 2.5. The Laplace functional of Π_t using the form of the Laplace functional of a Poisson process reads

$$\begin{aligned} & E \left[\exp - \int \phi(x) \Pi_t(dx) \right] \\ &= E \left[\exp - \int_{-\infty}^0 \left(1 - \mathbb{E} \left[\exp - \sum_{k \leq n(t)} \phi(x + x_k(t) - \sqrt{2}t) \right] \right) \sqrt{\frac{2}{\pi}} \left\{ -x e^{-\sqrt{2}x} \right\} dx \right] \\ &= E \exp \left[-\sqrt{\frac{2}{\pi}} \int_0^\infty u(t, x + \sqrt{2}t + \frac{1}{\sqrt{2}} \log Z) x e^{\sqrt{2}x} dx \right], \end{aligned} \quad (3.46)$$

with

$$u(t, x) = 1 - \mathbb{E} \left[\exp - \sum_{k=1}^{n(t)} \phi(-x + x_k(t)) \right].$$

By (3.25),

$$\lim_{t \rightarrow \infty} \sqrt{\frac{2}{\pi}} \int_0^\infty u(t, x + \sqrt{2}t + \frac{1}{\sqrt{2}} \log Z) x e^{\sqrt{2}x} dx = Z \sqrt{\frac{2}{\pi}} \lim_{t \rightarrow \infty} \int_0^\infty u(t, x + \sqrt{2}t) x e^{\sqrt{2}x} dx,$$

and the limit exists and is strictly positive by Proposition 3.7. This implies that the Laplace functionals of $\lim_{t \rightarrow \infty} \Pi_t$ and of the extremal process of BBM are equal. The proof of Theorem 2.5 is concluded. \square

3.5. Properties of the clusters. In this section we prove Proposition 2.8 and Proposition 2.9.

Proof of Proposition 2.8. Throughout the proof, the probabilities are considered conditional on Z . We show that for $\varepsilon > 0$ there exists C_1, C_2 such that

$$\sup_{t \geq t_0} P \left[\exists_{i,k} : \eta_i + \frac{1}{\sqrt{2}} \log Z + x_k^{(i)}(t) - \sqrt{2}t \geq Y, \text{ but } \eta_i \notin [-C_1\sqrt{t}, -C_2\sqrt{t}] \right] \leq \varepsilon. \quad (3.47)$$

The proof is split in two parts. We claim that, for t large enough, there exists $C_1 > 0$ small enough such that

$$P \left[\exists_{i,k} : \eta_i + \frac{1}{\sqrt{2}} \log Z + x_k^{(i)}(t) - \sqrt{2}t \geq Y, \text{ but } \eta_i \geq -C_1\sqrt{t} \right] \leq \varepsilon/2, \quad (3.48)$$

and $C_2 > 0$ large enough such that

$$P \left[\exists_{i,k} : \eta_i + \frac{1}{\sqrt{2}} \log Z + x_k^{(i)}(t) - \sqrt{2}t \geq Y, \text{ but } \eta_i \leq -C_2\sqrt{t} \right] \leq \varepsilon/2. \quad (3.49)$$

By Markov inequality the left-hand side of (3.48) is less than

$$\begin{aligned} & \int_{-C_1\sqrt{t}}^0 P \left[\max_k x_k(t) \geq \sqrt{2}t + Y - x - \frac{1}{\sqrt{2}} \log Z \right] (-x e^{-\sqrt{2}x}) dx \\ &= \int_0^{C_1\sqrt{t}} P \left[\max_k x_k(t) \geq \sqrt{2}t + Y + x - \frac{1}{\sqrt{2}} \log Z \right] x e^{\sqrt{2}x} dx \end{aligned} \quad (3.50)$$

This is the integral appearing in (3.43), truncated at $C_1\sqrt{t}$, and with δ replaced by Y . Hence, as $t \rightarrow \infty$,

$$(3.50) = \frac{e^{-\sqrt{2}Y} Z}{t^{3/2}} \int_{Y + \frac{3}{2\sqrt{2}} \log t}^{C_1\sqrt{t} + \frac{3}{2\sqrt{2}} \log t} x^2 \exp\left(-\frac{x^2}{2t} + \frac{3}{2\sqrt{2}} x \frac{\log t}{t}\right) dx + o(1). \quad (3.51)$$

By change of variable $x \rightarrow \frac{x}{\sqrt{t}}$ we see that, as $t \rightarrow \infty$ and up to irrelevant numerical constants,

$$(3.51) \leq Z e^{-\sqrt{2}Y} (1 + o(1)) \int_0^{C_1} x^2 e^{-x^2/2} dx. \quad (3.52)$$

But for smaller and smaller C_1 the integral is obviously vanishing: it thus suffices to choose C_1 small enough to have that (3.52) is less than $\varepsilon/2$, settling (3.48).

The proof of (3.49) is analogous, and we omit the details. The end result is that, for large enough t and up to irrelevant numerical constant,

$$(3.49) \leq Z e^{-\sqrt{2}Y} (1 + o(1)) \int_{C_2}^{\infty} x^2 e^{-x^2/2} dx. \quad (3.53)$$

It suffices to choose C_2 large enough to obtain (3.49). \square

For the proof of Proposition 2.9, the following Lemma is needed.

Lemma 3.10. *Let $u(t, x)$ be a solution to the F-KPP equation (3.3) with initial data satisfying*

$$\int_0^{\infty} y e^{\sqrt{2}y} u(0, y) dy < \infty,$$

and such that $u(t, \cdot + m(t))$ converges. Let ψ be the associated approximation as in Proposition 3.3. Then, for $x = a\sqrt{t}$ and $Y \in \mathbb{R}$,

$$\lim_{t \rightarrow \infty} \frac{e^{\sqrt{2}x} t^{3/2}}{x} \psi(r, t, x + Y + \sqrt{2}t) = K e^{-\sqrt{2}Y} e^{-a^2/2} \quad (3.54)$$

where

$$K = \sqrt{\frac{2}{\pi}} \int_0^{\infty} u(r, y' + \sqrt{2}r) y' e^{y' \sqrt{2}} dy'.$$

Moreover, the convergence is uniform for a in a compact set.

Proof. The proof is a simple computation:

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{e^{\sqrt{2}x}}{x} t^{3/2} \psi(r, t, Y + x + \sqrt{2}t) \\ & \stackrel{(3.4)}{=} e^{-\sqrt{2}Y} \lim_{t \rightarrow \infty} \frac{t^{3/2}}{x \sqrt{2\pi(t-r)}} \int_0^{\infty} u(r, y' + \sqrt{2}r) e^{y' \sqrt{2}} e^{-\frac{(y'-x-Y)^2}{2(t-r)}} (1 - e^{-2y' \frac{(x+Y + \frac{3}{2\sqrt{2}} \log t)}{t-r}}) dy' \\ & = e^{-\sqrt{2}Y} \int_0^{\infty} u(r, y' + \sqrt{2}r) e^{y' \sqrt{2}} \lim_{t \rightarrow \infty} \frac{t^{3/2}}{x \sqrt{2\pi(t-r)}} \left[e^{-\frac{(y'-x-Y)^2}{2(t-r)}} (1 - e^{-2y' \frac{(x+Y + \frac{3}{2\sqrt{2}} \log t)}{t-r}}) \right] dy', \end{aligned} \quad (3.55)$$

the last step by dominated convergence (cfr. (3.12)-(3.16)). Using that $x = a\sqrt{t}$,

$$\lim_{t \rightarrow \infty} \frac{t^{3/2}}{x\sqrt{2\pi}(t-r)} \left[e^{-\frac{(y'-x-Y)^2}{2(t-r)}} (1 - e^{-2y' \frac{(x+Y+\frac{3}{2\sqrt{2}} \log t)}{t-r}}) \right] = \sqrt{\frac{2}{\pi}} y' e^{-a^2/2}, \quad (3.56)$$

hence

$$\lim_{t \rightarrow \infty} \frac{e^{\sqrt{2}x}}{x} t^{3/2} \psi(r, t, Y + x + \sqrt{2}t) = K e^{-\sqrt{2}Y} e^{-a^2/2}. \quad (3.57)$$

□

Proof of Proposition 2.9. Let $a \in [-C_1, -C_2]$ and $b \in \mathbb{R}$. Set $x = a\sqrt{t} + b$. Define for convenience

$$\bar{\mathcal{E}}_t \equiv \sum_i \delta_{x_i(t) - \sqrt{2}t},$$

and $\max \bar{\mathcal{E}}_t \equiv \max_i x_i(t) - \sqrt{2}t$. We first claim that $x + \max \bar{\mathcal{E}}_t$ conditionally on $\{x + \max \bar{\mathcal{E}}_t > 0\}$ weakly converges to an exponential random variable,

$$\lim_{t \rightarrow \infty} \mathbb{P} \left[x + \max \bar{\mathcal{E}}_t > X \mid x + \max \bar{\mathcal{E}}_t > 0 \right] = e^{-\sqrt{2}X}, \quad (3.58)$$

for $X > 0$ (and 0 otherwise). Remark, in particular, that the limit does not depend on x .

To see (3.58), we write the conditional probability as

$$\frac{\mathbb{P} [x + \max \bar{\mathcal{E}}_t > X]}{\mathbb{P} [x + \max \bar{\mathcal{E}}_t > 0]}. \quad (3.59)$$

For t large enough (and hence $-x$ large enough in the positive) we may apply the uniform bounds from Proposition 3.3 in the form

$$\mathbb{P} \left[\max_{k \leq n(t)} x_k(t) \geq X - x + \sqrt{2}t \right] \leq \gamma(r) \psi(r, t, X - x + \sqrt{2}t) \quad (3.60)$$

and

$$\mathbb{P} \left[\max_{k \leq n(t)} x_k(t) \geq -x + \sqrt{2}t \right] \geq \gamma^{-1}(r) \psi(r, t, -x + \sqrt{2}t) \quad (3.61)$$

where ψ is as in (3.4) and the u entering into its definition is solution to F-KPP with Heaviside initial conditions, and r is large enough. Therefore,

$$\gamma^{-2}(r) \frac{\psi(r, t, X - x + \sqrt{2}t)}{\psi(r, t, -x + \sqrt{2}t)} \leq (3.59) \leq \gamma^2(r) \frac{\psi(r, t, X - x + \sqrt{2}t)}{\psi(r, t, -x + \sqrt{2}t)}. \quad (3.62)$$

By Lemma 3.10,

$$\lim_{t \rightarrow \infty} \frac{\psi(r, t, X - x + \sqrt{2}t)}{\psi(r, t, -x + \sqrt{2}t)} = e^{-\sqrt{2}X}. \quad (3.63)$$

Taking the limit $t \rightarrow \infty$ first, and then $r \rightarrow \infty$ (and using that $\gamma(r) \downarrow 1$) we thus see that (3.62) implies (3.58).

Second, we show that for any function ϕ that is continuous with compact support, the limit of

$$\mathbb{E} \left[\exp - \int \phi(x+z) \bar{\mathcal{E}}_t(dz) \mid x + \max \bar{\mathcal{E}}_t > 0 \right]$$

exists and is independent of x . It follows from the first part of the proof that the conditional process has a maximum almost surely. It is thus sufficient to consider the truncated Laplace functional, that is for $\delta > 0$,

$$\mathbb{E} \left[\exp \left(- \int \phi(x+z) \bar{\mathcal{E}}_t(dz) \right) \mathbb{1}_{\{x + \max \bar{\mathcal{E}}_t \leq \delta\}} \middle| x + \max \bar{\mathcal{E}}_t > 0 \right] \quad (3.64)$$

The above conditional expectation can be written as

$$\begin{aligned} &= \frac{\mathbb{E} \left[\prod_{k=1}^{n(t)} \exp \left(- \phi(x + x_k(t) - \sqrt{2}t) \right) \mathbb{1}_{\{x + x_k(t) - \sqrt{2}t \leq \delta\}} \right]}{\mathbb{P} \left[x + \max \bar{\mathcal{E}}_t > 0 \right]} \\ &\quad - \frac{\mathbb{E} \left[\prod_{k=1}^{n(t)} \exp \left(- \phi(x + x_k(t) - \sqrt{2}t) \right) \mathbb{1}_{\{x + x_k(t) - \sqrt{2}t \leq 0\}} \right]}{\mathbb{P} \left[x + \max \bar{\mathcal{E}}_t > 0 \right]}. \end{aligned} \quad (3.65)$$

Define

$$\begin{aligned} u_1(t, y) &\equiv 1 - \mathbb{E} \left[\prod_{k=1}^{n(t)} e^{-\phi(-y + x_k(t))} \mathbb{1}_{\{-y + x_k(t) \leq 0\}} \right] \\ u_2(t, y) &\equiv 1 - \mathbb{E} \left[\prod_{k=1}^{n(t)} e^{-\phi(-y + x_k(t))} \mathbb{1}_{\{-y + x_k(t) \leq \delta\}} \right] \\ u_3(t, y) &\equiv \mathbb{P} \left[-y + \max_k x_k(t) \leq 0 \right] \end{aligned}$$

so that

$$(3.65) = \frac{u_2(t, -x + \sqrt{2}t)}{u_3(t, -x + \sqrt{2}t)} - \frac{u_1(t, -x + \sqrt{2}t)}{u_3(t, -x + \sqrt{2}t)}. \quad (3.66)$$

Remark that the functions u_1, u_2 and u_3 , all solve the F-KPP equation (3.3) with initial conditions

$$\begin{aligned} u_1(0, y) &= 1 - e^{-\phi(-y)} \mathbb{1}_{\{-y \leq 0\}}, \\ u_2(0, y) &= 1 - e^{-\phi(-y)} \mathbb{1}_{\{-y \leq \delta\}}, \\ u_3(0, y) &= 1 - \mathbb{1}_{\{-y \leq 0\}}. \end{aligned} \quad (3.67)$$

They also satisfy the assumptions of Theorem 3.2 and Proposition 3.3.

Let ψ_i be as in (3.4) with u replaced by the appropriate u_i , $i = 1, 2, 3$. By Proposition 3.3,

$$\begin{aligned} &\lim_{t \rightarrow \infty} \frac{u_2(t, -x + \sqrt{2}t)}{u_3(t, -x + \sqrt{2}t)} - \frac{u_1(t, -x + \sqrt{2}t)}{u_3(t, -x + \sqrt{2}t)} \\ &= \lim_{r \rightarrow \infty} \lim_{t \rightarrow \infty} \left\{ \frac{\psi_2(r, t, -x + \sqrt{2}t)}{\psi_3(r, t, -x + \sqrt{2}t)} \right\} - \lim_{r \rightarrow \infty} \lim_{t \rightarrow \infty} \left\{ \frac{\psi_1(r, t, -x + \sqrt{2}t)}{\psi_3(r, t, -x + \sqrt{2}t)} \right\}. \end{aligned} \quad (3.68)$$

By Lemma 3.10, the above limits exist and do not depend on x . This shows the existence of (3.64). It remains to prove that this limit is non-zero for a non-trivial ϕ , thereby

showing the existence and local finiteness of the conditional point process. To see this, note that, by Proposition 3.4, the limit of (3.64) equals

$$\frac{C_2 - C_1}{C_3}$$

where $C_i = \lim_{t \rightarrow \infty} \int_{-\infty}^0 u_i(t, y' + \sqrt{2}t)(-y'e^{-\sqrt{2}y'} dy')$ for u_i as above. Note that $0 < C_3 < \infty$, since by the representation of Theorem 2.2

$$\lim_{t \rightarrow \infty} \mathbb{P}(\max x_i(t) - m(t) \leq z) = \mathbb{E} \exp -C_3 Z e^{-\sqrt{2}z},$$

and this probability is non-trivial. Now suppose $C_1 = C_2$. Then by Theorem 2.2 again, this would entail

$$\lim_{t \rightarrow \infty} \mathbb{E} \left[\left(\exp - \int \phi(x) \mathcal{E}_t(dx) \right) \mathbb{1}_{\{\max \mathcal{E}_t < \delta\}} \right] = \lim_{t \rightarrow \infty} \mathbb{E} \left[\left(\exp - \int \phi(x) \mathcal{E}_t(dx) \right) \mathbb{1}_{\{\max \mathcal{E}_t < 0\}} \right],$$

where $\mathcal{E}_t = \sum_i \delta_{x_i(t) - m(t)}$ and $\max \mathcal{E}_t = \max_i x_i(t) - m(t)$. Thus,

$$\lim_{t \rightarrow \infty} \mathbb{E} \left[\left(\exp - \int \phi(x) \mathcal{E}_t(dx) \right) \mathbb{1}_{\{0 < \max \mathcal{E}_t < \delta\}} \right] = 0.$$

But this is impossible since the maximum has positive probability of occurrence in $[0, \delta]$ for any δ and the process $\lim_{t \rightarrow \infty} \mathcal{E}_t$ is locally finite. This concludes the proof of the Proposition. \square

Define the gap process at time t

$$\mathcal{D}_t \equiv \sum_i \delta_{x_i(t) - \max_j x_j(t)}. \quad (3.69)$$

Let us write $\bar{\mathcal{E}}$ for the point process obtained in Proposition 2.9 from the limit of the conditional law of $\bar{\mathcal{E}}_t$ given $\max \bar{\mathcal{E}}_t > 0$. We denote by $\max \bar{\mathcal{E}}$ the maximum of $\bar{\mathcal{E}}$, and by \mathcal{D} the process of the gaps of $\bar{\mathcal{E}}$, that is the process $\bar{\mathcal{E}}$ shifted back by $\max \bar{\mathcal{E}}$. The following corollary is the fundamental result showing that \mathcal{D} is the limit of the conditioned process \mathcal{D}_t , and, perhaps surprisingly, the process of the gaps in the limit is independent of the location of the maximum.

Corollary 3.11. *Let $x = a\sqrt{t}$, $a < 0$. In the limit $t \rightarrow \infty$, the random variables \mathcal{D}_t and $x + \max \bar{\mathcal{E}}$ are conditionally independent on the event $x + \max \bar{\mathcal{E}} > b$ for any $b \in \mathbb{R}$. More precisely, for any bounded continuous function f, h and $\phi \in \mathcal{C}_c(\mathbb{R})$,*

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathbb{E} \left[f \left(\int \phi(z) \mathcal{D}_t(dz) \right) h(x + \max \bar{\mathcal{E}}_t) \middle| x + \max \bar{\mathcal{E}}_t > b \right] \\ = \mathbb{E} \left[f \left(\int \phi(z) \mathcal{D}(dz) \right) \right] \int_b^\infty h(y) \frac{\sqrt{2}e^{-\sqrt{2}y} dy}{e^{-\sqrt{2}b}}. \end{aligned}$$

Moreover, convergence is uniform in $x = a\sqrt{t}$ for a in a compact set.

Proof. By standard approximation, it suffices to establish the result for $h(y) = \mathbb{1}_{\{y > b'\}}$ for $b' > b$. By the property of conditioning and since $b' > b$

$$\begin{aligned} & \mathbb{E} \left[f \left(\int \phi(z) \mathcal{D}_t(dz) \right) \mathbb{1}_{\{x + \max \mathcal{E}_t > b'\}} \middle| x + \max \bar{\mathcal{E}}_t > b \right] \\ &= \mathbb{E} \left[f \left(\int \phi(z) \mathcal{D}_t(dz) \right) \middle| x - b' + \max \bar{\mathcal{E}}_t > 0 \right] \frac{\mathbb{P} [x - b + \max \bar{\mathcal{E}}_t > b' - b]}{\mathbb{P} [x - b + \max \bar{\mathcal{E}}_t > 0]} . \end{aligned}$$

The conclusion will follow from Proposition 2.9 by taking the limit $t \rightarrow \infty$, once it is shown that convergence of $(\bar{\mathcal{E}}_t, y + \max \bar{\mathcal{E}}_t)$ under the conditional law implies convergence of the gap process \mathcal{D}_t . This is a general continuity result which is done in the next lemma. \square

Lemma 3.12. *Let (μ_t, X_t) be a sequence of random variables on $\mathcal{M} \times \mathbb{R}$ that converges to (μ, X) in the sense that for any bounded continuous function f, h on \mathbb{R} and any $\phi \in \mathcal{C}_c(\mathbb{R})$*

$$\mathbb{E} \left[f \left(\int \phi d\mu_t \right) h(X_t) \right] \rightarrow \mathbb{E} \left[f \left(\int \phi d\mu \right) h(X) \right] .$$

Then for any $\phi \in \mathcal{C}_c(\mathbb{R})$ and $g : \mathbb{R} \rightarrow \mathbb{R}$, bounded continuous,

$$\mathbb{E} \left[g \left(\int \phi(y + X_t) \mu_t(dy) \right) \right] \rightarrow \mathbb{E} \left[g \left(\int \phi(y + X) \mu(dy) \right) \right] .$$

Proof. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded continuous function. Introduce the notation

$$T_x \mu(\phi) \equiv \int \phi(y + x) \mu(dy) .$$

We need to show that for t large enough

$$\left| \mathbb{E} [f(T_{X_t} \mu_t(\phi))] - \mathbb{E} [f(T_X \mu(\phi))] \right| \tag{3.70}$$

is smaller than ε . By standard approximations, it is enough to suppose f is Lipschitz, whose constant we assume to be 1 for simplicity.

Since the random variables (X_t) are tight by assumption, there exist $t(\varepsilon)$ large enough and K_ε , an interval of \mathbb{R} , such that

$$(3.70) \leq \left| \mathbb{E} [f(T_{X_t} \mu_t(\phi)); X_t \in K_\varepsilon] - \mathbb{E} [f(T_X \mu(\phi)); X \in K_\varepsilon] \right| + \varepsilon .$$

Now divide K_ε into N intervals I_j of equal length. Write \bar{x}_j for the midpoint of I_j . For each of these intervals, one has

$$\mathbb{E} [f(T_{X_t} \mu_t(\phi)); X_t \in I_j] = \mathbb{E} [f(T_{\bar{x}_j} \mu_t(\phi)); X_t \in I_j] + \mathcal{R}(t, j) \tag{3.71}$$

for

$$\mathcal{R}(t, j) \leq \mathbb{E} [|f(T_{X_t} \mu_t(\phi)) - f(T_{\bar{x}_j} \mu_t(\phi))|; X_t \in I_j]$$

Since f is Lipschitz, the right-hand side is smaller than

$$\mathbb{E} [|T_{X_t} \mu_t(\phi) - T_{\bar{x}_j} \mu_t(\phi)|; X_t \in I_j] . \tag{3.72}$$

Moreover

$$|T_{X_t} \mu_t(\phi) - T_{\bar{x}_j} \mu_t(\phi)| = \int |\phi(y - X_t) - \phi(y - \bar{x}_j)| \mu_t(dy) .$$

Note that there exists a compact C , independently of t and j so that $|\phi(y - X_t) - \phi(y - \bar{x}_j)| = 0$ for $y \notin C$. (It suffices to take C so that it contains all the translates $\text{supp}\phi + k$, $k \in K_\varepsilon$). By taking N large enough, $|y - X_t - (y - \bar{x}_j)| = |\bar{x}_j - X_t| < \delta_\phi$ for the appropriate δ_ϕ making $|\phi(y - X_t) - \phi(y - \bar{x}_j)| < \varepsilon$, uniformly on $y \in C$. Hence, (3.72) is smaller than

$$\varepsilon \mathbb{E}[\mu_t(C); X_t \in I_j]$$

The summation over j is thus smaller than $\varepsilon \mathbb{E}[\mu_t(C)]$. By the convergence of (μ_t) , this can be made smaller for t large enough.

The same approximation scheme for (μ, X) yields

$$\mathbb{E}[f(T_X \mu(\phi)); X_t \in I_j] = \mathbb{E}[f(T_{\bar{x}_j} \mu(\phi)); X \in I_j] + \mathcal{R}(j) \quad (3.73)$$

where $\sum_j \mathcal{R}(j) \leq \varepsilon \mathbb{E}[\mu(C)]$. Therefore (3.70) will hold provided that the difference of the first terms of the right-hand side of (3.71) and of (3.73) is small for t large enough and N fixed. But this is guaranteed by the hypotheses on the convergence of (μ_t, X_t) . \square

3.6. Characterisation of the extremal process.

Proof of Theorem 2.1. It suffices to show that for $\phi : \mathbb{R} \rightarrow \mathbb{R}_+$ continuous with compact support the Laplace functional of the extremal process of branching Brownian motion satisfies

$$\lim_{t \rightarrow \infty} \Psi_t(\phi) = \mathbb{E} \exp \left(-CZ \int_{\mathbb{R}} \mathbb{E}[1 - e^{-\int \phi(y+z) \mathcal{D}(dz)}] \sqrt{2} e^{-\sqrt{2}y} dy \right) \quad (3.74)$$

for the point process \mathcal{D} of Corollary 3.11.

Now, by Theorem 2.5,

$$\lim_{t \rightarrow \infty} \Psi_t(\phi) = \lim_{t \rightarrow \infty} \mathbb{E} \left[\exp - \sum_{i,k} \phi(\eta_i + \frac{1}{\sqrt{2}} \log Z + x_k^{(i)}(t) - \sqrt{2}t) \right]. \quad (3.75)$$

Using the form for the Laplace transform of a Poisson process we have for the r.h.s. above

$$\begin{aligned} & \lim_{t \rightarrow \infty} \mathbb{E} \left[\exp - \sum_{i,k} \phi(\eta_i + \frac{1}{\sqrt{2}} \log Z + x_k^{(i)}(t) - \sqrt{2}t) \right] \\ &= \mathbb{E} \exp \left(-Z \lim_{t \rightarrow \infty} \int_{-\infty}^0 \mathbb{E} \left[1 - \exp - \int \phi(x+y) \bar{\mathcal{E}}_t(dx) \right] \sqrt{\frac{2}{\pi}} (-ye^{-\sqrt{2}y}) dy \right). \end{aligned} \quad (3.76)$$

Let \mathcal{D}_t as in (3.69). The integral of the right-hand side above can be written as

$$\lim_{t \rightarrow \infty} \int_{-\infty}^0 \mathbb{E} \left[f \left(\int \{T_{y+\max} \bar{\mathcal{E}}_t \phi(z)\} \mathcal{D}_t(dz) \right) \right] \sqrt{\frac{2}{\pi}} (-ye^{-\sqrt{2}y}) dy$$

for the bounded (on $[0, \infty)$) continuous function $f(x) = 1 - e^{-x}$, and where $T_x\phi(y) = \phi(y + x)$. By Proposition 2.8, there exist C_1 and C_2 such that

$$\begin{aligned} & \int_{-\infty}^0 \mathbb{E} \left[f \left(\int \{T_{y+\max \bar{\mathcal{E}}_t} \phi(z)\} \mathcal{D}_t(dz) \right) \right] \sqrt{\frac{2}{\pi}} (-ye^{-\sqrt{2}y}) dy \\ &= \Omega_t(C_1, C_2) + \int_{-C_2\sqrt{t}}^{-C_1\sqrt{t}} \mathbb{E} \left[f \left(\int \{T_{y+\max \bar{\mathcal{E}}_t} \phi(z)\} \mathcal{D}_t(dz) \right) \right] \sqrt{\frac{2}{\pi}} (-ye^{-\sqrt{2}y}) dy, \end{aligned} \quad (3.77)$$

where the error term satisfies $\lim_{C_1 \downarrow 0, C_2 \uparrow \infty} \sup_{t \geq t_0} \Omega_t(C_1, C_2) = 0$. Introducing a conditioning on the event $y + \max \bar{\mathcal{E}}_t > m_\phi$, the term in the integral becomes

$$\begin{aligned} & \mathbb{E} \left[f \left(\int \{T_{y+\max \bar{\mathcal{E}}_t} \phi(z)\} \mathcal{D}_t(dz) \right) \right] = \\ & \mathbb{E} \left[f \left(\int \{T_{y+\max \bar{\mathcal{E}}_t} \phi(z)\} \mathcal{D}_t(dz) \right) \middle| y + \max \bar{\mathcal{E}}_t > m_\phi \right] \mathbb{P} [y + \max \bar{\mathcal{E}}_t > m_\phi] . \end{aligned} \quad (3.78)$$

By Corollary 3.11, the conditional law of the pair $\mathcal{D}_t, y + \max \bar{\mathcal{E}}_t$ given $\{y + \max \bar{\mathcal{E}}_t > m_\phi\}$ exists in the limit. Moreover the convergence is uniform in $y \in [-C_1\sqrt{t}, -C_2\sqrt{t}]$. By Lemma 3.12, the convergence applies to the random variable $\int \{T_{y+\max \bar{\mathcal{E}}_t} \phi(z)\} \mathcal{D}_t(dz)$. Therefore

$$\begin{aligned} & \lim_{t \rightarrow \infty} \mathbb{E} \left[f \left(\int \{T_{y+\max \bar{\mathcal{E}}_t} \phi(z)\} \mathcal{D}_t(dz) \right) \middle| y + \max \bar{\mathcal{E}}_t > m_\phi \right] \\ &= \int_{m_\phi}^{\infty} \mathbb{E} \left[f \left(\int (T_y \phi(z)) \mathcal{D}(dz) \right) \right] \frac{\sqrt{2}e^{-\sqrt{2}y} dy}{e^{-\sqrt{2}m_\phi}} \end{aligned} \quad (3.79)$$

On the other hand,

$$\int_{-C_2\sqrt{t}}^{-C_1\sqrt{t}} \mathbb{P} [y + \max \bar{\mathcal{E}}_t > m_\phi] \sqrt{\frac{2}{\pi}} (-ye^{-\sqrt{2}y}) dy = Ce^{-\sqrt{2}m_\phi} + \Omega_t(C_1, C_2) \quad (3.80)$$

by Lemma 3.6 and by the same approximation as in (3.77).

Combining (3.80), (3.79) and (3.78) gives that (3.76) converges to

$$\mathbb{E} \exp \left(-CZ \int_{\mathbb{R}} \mathbb{E}[1 - e^{-\int \phi(y+z) \mathcal{D}(dz)}] \sqrt{2}e^{-\sqrt{2}y} dy \right) ,$$

which is by (3.75) also the limiting Laplace transform of the extremal process of branching Brownian motion: this shows (3.74) and thus concludes the proof of Theorem 2.1. \square

REFERENCES

- [1] M. Aizenman and L.-P. Arguin, *On the Structure of Quasi-Stationary Competing Particle Systems*, Ann. Probab. 37, no. 3, pp. 1080-1113 (2009)
- [2] M. Aizenman, R. Sims, and S. Starr, *Mean Field Spin Glass Models from the Cavity-ROSt Perspective*, In *Prospects in Mathematical Physics* AMS Contemporary Mathematics, vol. 437 (2007)
- [3] L.-P. Arguin, A. Bovier, and N. Kistler, *The genealogy of extremal particles in branching Brownian Motion*, [arXiv:1008.4386](#)

- [4] L.-P. Arguin, A. Bovier, and N. Kistler, *Poissonian Statistics in the Extremal Process of Branching Brownian Motion*, [arXiv:1010.2376](#)
- [5] D.G. Aronson and H.F. Weinberger, *Nonlinear diffusion in population genetics, combustion and nerve propagation*, in *Partial Differential Equations and Related Topics*, ed. J.A. Goldstein; Lecture Notes in Mathematics No. 446, pp.5-49, Springer, New York (1975)
- [6] D.G. Aronson and H.F. Weinberger, *Multi-dimensional nonlinear diffusions arising in population genetics*, *Adv. in Math.*, 30, no. 1, pp. 33–76 (1978)
- [7] E. Bolthausen, J.-D. Deuschel, and G. Giacomin, *Entropic repulsion and the maximum of the two-dimensional harmonic crystal*, *Ann. Probab.* 29 (2001)
- [8] E. Bolthausen, J.-D. Deuschel, and O. Zeitouni, *Recursions and tightness for the maximum of the discrete, two dimensional Gaussian Free Field*, [arXiv:1005.5417](#)
- [9] A. Bovier and I. Kurkova *Derrida’s Generalized Random Energy Models. 1. Models with finitely many hierarchies*. *Ann. Inst. H. Poincaré. Prob. et Statistiques (B) Prob. Stat.* 40, pp. 439-480 (2004)
- [10] A. Bovier, *Statistical mechanics of disordered systems. A mathematical perspective*, Cambridge University Press, Cambridge (2005).
- [11] A. Bovier and I. Kurkova *Derrida’s generalized random energy models. 2. Models with continuous hierarchies*. *Ann. Inst. H. Poincaré. Prob. et Statistiques (B) Prob. Stat.* 40, pp. 481-495 (2004)
- [12] M. Bramson, *Maximal displacement of branching Brownian motion*, *Comm. Pure Appl. Math.* 31, pp. 531-581 (1978)
- [13] M. Bramson, *Convergence of solutions of the Kolmogorov equation to travelling waves*, *Mem. Amer. Math. Soc.* 44, no. 285, iv+190 pp. (1983)
- [14] M. Bramson and O. Zeitouni, *Tightness of the recentered maximum of the two-dimensional discrete Gaussian Free Field*, [arXiv:1009.3443](#)
- [15] B. Chauvin and A. Rouault, *Supercritical Branching Motion and K-P-P Equation in the Critical Speed-Area*, *Math. Nachr.* 149 (1990).
- [16] B. Chauvin and A. Rouault, *KPP equation and supercritical branching Brownian motion in the subcritical speed area. Application to spatial trees*, *Prob. Theor. Rel. Fields* 80, pp. 299-314 (1988)
- [17] B. Derrida, *A generalization of the random energy model that includes correlations between the energies*, *J.Phys.Lett.* **46**, pp. 401-407 (1985)
- [18] E. Brunet and B. Derrida, *Statistics at the tip of a branching random walk and the delay of traveling waves*, *Europhys. Lett.* 87: 60010, 5 pp. (2009)
- [19] E. Brunet and B. Derrida, *A branching random walk seen from the tip*, [arXiv:1011.4864](#) (2010)
- [20] A. Dembo, *Simple random covering, disconnection, late and favorite points*, *Proceedings of the International Congress of Mathematicians, Madrid, Volume III*, pp. 535-558 (2006)
- [21] A. Dembo, Y. Peres, J. Rosen, and O. Zeitouni, *Cover times for Brownian motion and random walks in two dimensions*, *Ann. Math.*, 160 pp. 433-464 (2004)
- [22] B. Derrida and H. Spohn, *Polymers on disordered trees, spin glasses, and travelling waves*, *J. Statist. Phys.* 51, no. 5-6, pp. 817-840 (1988)
- [23] R.A. Fisher, *The wave of advance of advantageous genes*, *Ann. Eugen.* 7: 355-369 (1937)
- [24] S.C. Harris, *Travelling-waves for the FKPP equation via probabilistic arguments*, *Proc. Roy. Soc. Edin.*, 129A, pp. 503-517 (1999)
- [25] O. Kallenberg, *Random Measures*, Springer (1983), 187 pp.
- [26] A. Kolmogorov, I. Petrovsky, and N. Piscounov, *Etude de l’équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique*, *Moscov Universitet, Bull. Math.* 1, pp. 1-25 (1937)
- [27] S.P. Lalley and T. Sellke, *A conditional limit theorem for the frontier of a branching brownian motion*, *Ann. Probab.* 15, no.3, pp. 1052-1061 (1987)
- [28] M.R. Leadbetter, G. Lindgren and H. Rootzen, *Extremes and related properties of random sequences and processes*, Springer Series in Statistics. Springer-Verlag, New York-Berlin (1983)
- [29] H.P. McKean, *Application of Brownian Motion to the equation of Kolmogorov-Petrovskii-Piskunov*, *Comm.Pure. Appl. Math.* 28, pp. 323-331 (1976)

- [30] M. Mézard, G. Parisi, and M. Virasoro, *Spin Glass theory and beyond*, World scientific, Singapore (1987)
- [31] A. Ruzmaikina and M. Aizenman, *Characterization of invariant measures at the leading edge for competing particle systems*, Ann. Probab. 33, no. 1, 82–113 (2005)
- [32] T. H. Scheike, *A Boundary-Crossing Result for Brownian Motion*, Journal of Applied Probability, Vol. 29, No. 2, pp. 448-453 (1992)
- [33] M. Talagrand, *Spin Glasses: A Challenge for Mathematicians. Cavity and Mean Field Models*, Springer Verlag (2003)

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